Rational Ehrhart Theory

Sophie Rehberg
joint work with Matthias Beck & Sophia Elia

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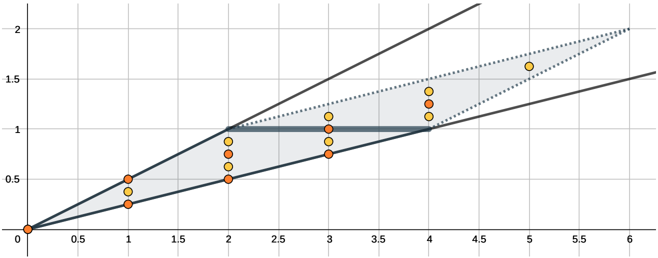


Ehrhart (Quasi-)Polynomials

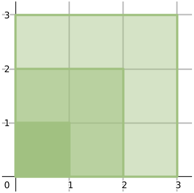
P a d-polytope in $\mathbb{R}^d$, $n \in \mathbb{Z}_{>0}$.

$$n\mathrm{P} := \{n\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in \mathrm{P}\}$$

$$\mathrm{ehr}_{\mathrm{P}}(n) := \#\left(\mathbb{Z}^d \cap n\mathrm{P}\right)$$

Example: unit square $[0,1]^2$

$$\mathrm{ehr}\left([0,1]^2; n\right) = (n+1)^2$$



Theorem (Ehrhart 1962)
P an integral d-polytope. Then $\mathrm{ehr}(\mathrm{P}; n)$ agrees with a polynomial of degree d.

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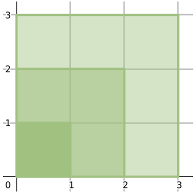


Example: unit square $[0,1]^2$

$$\mathrm{ehr}\left([0,1]^2; n\right) = (n+1)^2$$

Theorem (Ehrhart 1962)
P a ~~integral~~ rational d-polytope. Then, $\mathrm{ehr}(\mathrm{P}; n)$ agrees with a quasipolynomial of degree d, period $p \mid k$.

quasipolynomial: $q(x) = c_0(x) + c_1(x)x + \cdots + c_d(x)x^d$,
$c_i(x)$ periodic functions
denominator k of P is lcm of denominators of coordinates of vertices

Ehrhart Series

Theorem (Stanley 1980)
$\mathrm{P} \subset \mathbb{R}^d$ a rational d-polytope with denominator k. Then

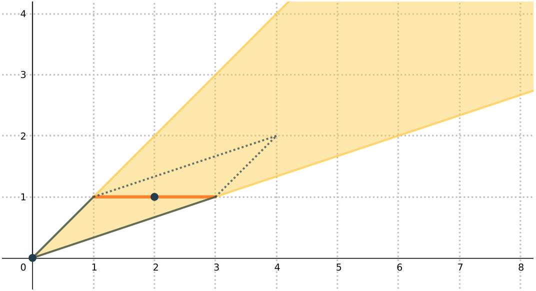
$$\mathrm{Ehr}(\mathrm{P};t) := 1 + \sum_{n \geq 1} \mathrm{ehr}(\mathrm{P};n)\, t^n = \frac{\mathrm{h}^*(\mathrm{P};t)}{(1-t^k)^{d+1}}$$

and $\mathrm{h}^*(\mathrm{P};t)$ is a polynomial with nonnegative coefficients.

Example:

$\mathrm{P}_3 := [1,3]$

$$\mathrm{Ehr}(\mathrm{P}_3;t) = \frac{1+t}{(1-t)^2}$$



Literature

Linke (2011) $|\lambda P \cap \mathbb{Z}^d|$ for $\lambda \in \mathbb{Q}_{>0}$ is quasipolynomial, coefficients are piece-wise polynomial and related by derivatives,...

Baldoni-Berline-Köppe-Vergne (2013) intermediate sums on polyhedra, with $|\lambda P \cap \mathbb{Z}^d|$ as special case.

Stapledon (2008,2017) introduced weighted h*-polynomials, investigated a Ehrhart series counting points on boundaries for polytopes with $\mathbf{0} \in P$

For $\mathsf{P} \subset \mathbb{R}^d$ define the **rational Ehrhart counting function** as

$$\operatorname{rehr}(\mathsf{P};\lambda) := |\lambda \mathsf{P} \cap \mathbb{Z}^d| \quad \text{for } \lambda \in \mathbb{Q}_{>0}\,.$$

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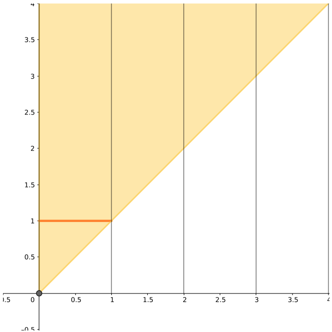
Examples:

$\mathsf{P}_1 = [0,1] \subset \mathbb{R}$

$$\mathrm{rehr}(\mathsf{P}_1;\lambda) \;=\; \lfloor \lambda \rfloor + 1$$



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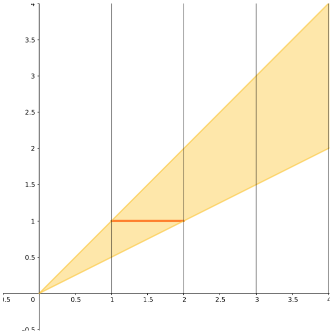
$$\operatorname{rehr}(P;\lambda) := |\lambda P \cap \mathbb{Z}^d| \quad \text{for } \lambda \in \mathbb{Q}_{>0}\,.$$

Examples:

$P_2 = [1,2] \subset \mathbb{R}$

$$\operatorname{rehr}(P_2;\lambda) = \lfloor 2\lambda \rfloor - \lceil \lambda \rceil + 1$$



Let

$$\mathrm{P} = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{A}\mathbf{x} \leq \mathbf{b} \right\}$$

with $\mathbf{A} \in \mathbb{Z}^{n \times d}$, $\mathbf{b} \in \mathbb{Z}^n$ and every row is in lowest terms.
We define the **codenominator** r:

$$r = \operatorname{lcm}(\mathbf{b})\,.$$

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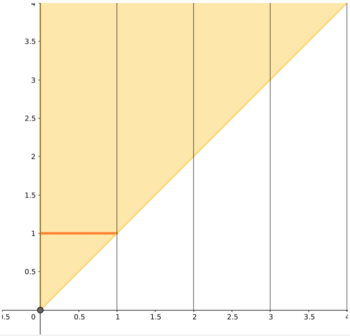
Examples:

$$\begin{aligned} P_1 &= [0,1] \\ &= \{x \in \mathbb{R} : \quad -x \le 0, \\ &\qquad\qquad\qquad\quad x \le 1\} \end{aligned}$$

so $r = 1$.



Set up

Definitions and Examples

Let

$$P = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{A}\mathbf{x} \leq \mathbf{b} \right\}$$

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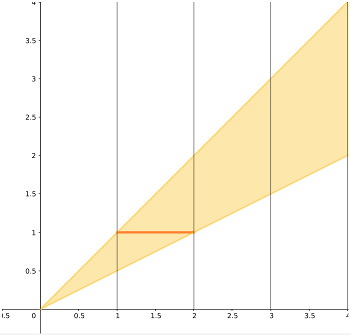
We define the **codenominator** r:

$$r = \operatorname{lcm}(\mathbf{b}) .$$

Examples:

$$\begin{aligned} P_2 &= [1,2] \\ &= \{x \in \mathbb{R} \ : \quad x \leq 2, \\ &\qquad\qquad\quad -x \leq -1\} \end{aligned}$$

so $r = 2$.



Discretizing counting

Proposition

Let $\mathsf{P} \subset \mathbb{R}^d$ be a rational d-polytope with codenominator r. Then

1. $\mathrm{rehr}(\mathsf{P};\lambda)$ is constant for $\lambda \in \left(\frac{n}{r}, \frac{n+1}{r}\right)$, $n \in \mathbb{Z}_{\geq 0}$.
2. If $\mathbf{0} \in \mathsf{P}$, then $\mathrm{rehr}(\mathsf{P};\lambda)$ is monotone and constant for $\lambda \in [\frac{n}{r}, \frac{n+1}{r})$, $n \in \mathbb{Z}_{\geq 0}$.

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We define the **(refined) rational Ehrhart series** as

$$\operatorname{REhr}(\mathsf{P};t) := 1 + \sum_{n \in \mathbb{Z}_{\geq 1}} \operatorname{rehr}\left(\mathsf{P}; \frac{n}{r}\right) t^{\frac{n}{r}}$$

$$\operatorname{RREhr}(\mathsf{P};t) := 1 + \sum_{n \in \mathbb{Z}_{\geq 1}} \operatorname{rehr}\left(\mathsf{P}; \frac{n}{2r}\right) t^{\frac{n}{2r}}$$

Rational Ehrhart Series and Generating Functions

Recall the **(refined) rational Ehrhart series** as

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Theorem

Let P be a rational d-polytope with codenominator r, and let $m \in \mathbb{Z}_{>0}$ such that $\frac{m}{r}\mathsf{P}$ is a lattice polytope. Then

$$\mathrm{REhr}\,(\mathsf{P};t) = \frac{\mathrm{rh}^*_m\,(\mathsf{P};t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}$$

where $\mathrm{rh}^*\,(\mathsf{P};t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

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Rational Ehrhart Series and Generating Functions

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where $\mathrm{rh}^*(\mathrm{P};t)$ is a polynomial in $\mathbb{Z}\left[t^{\frac{1}{r}}\right]$ with nonnegative coefficients.

Proof:

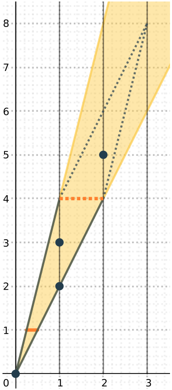
$$\mathrm{REhr}(\mathrm{P};t) = 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{rehr}\left(\mathrm{P};\frac{n}{r}\right) t^{\frac{n}{r}}$$

$$= 1 + \sum_{n \in \mathbb{Z}_{>0}} \mathrm{ehr}\left(\frac{1}{r}\mathrm{P};n\right)\left(t^{\frac{1}{r}}\right)^n = \frac{\mathrm{h}^*\left(\frac{1}{r}\mathrm{P};t^{\frac{1}{r}}\right)}{\left(1 - t^{\frac{m}{r}}\right)^{d+1}}.$$

□

Recall:

$$\mathrm{RREhr}(\mathrm{P}, t) = \frac{\mathrm{rrh}^*_m(\mathrm{P}; t)}{\left(1 - t^{\frac{m}{2r}}\right)^{d+1}}.$$



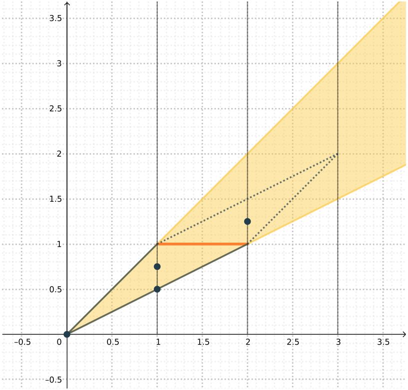
$\mathrm{P}_2 = [1, 2]$
$r = 2$
$\frac{1}{4}\mathrm{P}_2 = [\frac{1}{4}, \frac{1}{2}]$
$m = 4$
so $\frac{m}{2r} = 1$

$$\mathrm{RREhr}(\mathrm{P}_2; t) = \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2}$$

Example (continued)

Recall:

$$\mathrm{RREhr}(\mathrm{P}, t) = \frac{\mathrm{rrh}^*_m(\mathrm{P}; t)}{\left(1 - t^{\frac{m}{2r}}\right)^{d+1}} .$$



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Corollaries

Recall Theorem

P rational, codenominator r, $m \in \mathbb{Z}_{>0}$ s. t. $\frac{m}{r}\mathrm{P}$ is lattice

$$\mathrm{REhr}(\mathrm{P};t) = \frac{\mathrm{rh}^*_m(\mathrm{P};t)}{\left(1-t^{\frac{m}{r}}\right)^{d+1}}, \qquad \mathrm{rh}^*_m(\mathrm{P};t) \in \mathbb{Z}_{\geq 0}[t^{\frac{1}{r}}].$$

Corollaries

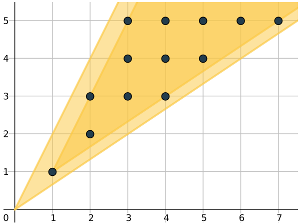
1. Period (Linke 2011): $\mathrm{rehr}(\mathrm{P};\lambda)$ is a quasipolynomial with period $\frac{j}{r}$ where $j \mid m$.
2. Reciprocity (Linke 2011): $(-1)^d\,\mathrm{rehr}(\mathrm{P};-\lambda) = |\lambda\mathrm{P}^\circ \cap \mathbb{Z}^d|$
3. If $\frac{m}{r} \in \mathbb{Z}$ we can retrieve the h*-polynomial from rh^*_m by extracting the terms with integer powers.

$$\mathrm{RREhr}(\mathrm{P}_2;t) = \frac{1+t^{\frac{1}{2}}+t^{\frac{3}{4}}+t^{\frac{5}{4}}}{(1-t)^2} \quad \to \quad \mathrm{Ehr}(\mathrm{P}_2;t) = \frac{1}{(1-t)^2}$$

Gorenstein polytopes

$\mathsf{C} \subset \mathbb{R}^{d+1}$ a pointed, rational, $(d+1)$-cone is called a **Gorenstein cone** if there is a **Gorenstein point** $(g, \mathbf{y}) \in \mathbb{Z}^{d+1}$ s.t.

$$\mathsf{C}^\circ \cap \mathbb{Z}^{d+1} = ((g, \mathbf{y}) + \mathsf{C}) \cap \mathbb{Z}^{d+1} .$$



A lattice polytope $\mathsf{P} \subset \mathbb{R}^d$ is called a **Gorenstein polytope** if $\hom(\mathsf{P})$ is a Gorenstein cone.

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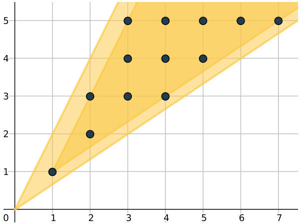
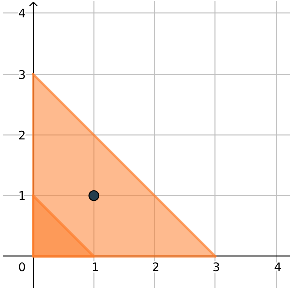


A lattice polytope $\mathsf{P} \subset \mathbb{R}^d$ is called a **Gorenstein polytope** if $\hom(\mathsf{P})$ is a Gorenstein cone.

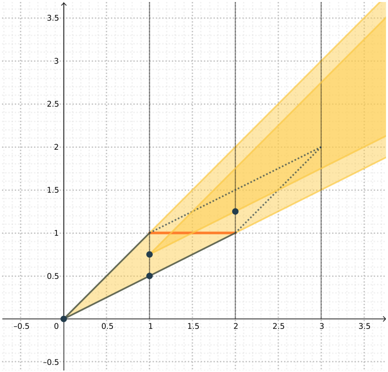
Nice properties, e.g.,

- $g\mathsf{P}$ has unique interior lattice point, for some $g \in \mathbb{Z}_{>0}$
- palindromic h^*-polynomial



A rational polytope $\mathrm{P} \subset \mathbb{R}^d$ is called γ-**rational Gorenstein** if $\hom\left(\frac{1}{\gamma}\mathrm{P}\right)$ is a Gorenstein cone.

A lattice polytope P is Gorenstein $\Leftrightarrow$ it is 1-rational Gorenstein.



$\mathrm{P}_2 = [1, 2]$, $r = 2$,
$\frac{1}{4}\mathrm{P}_2 = [\frac{1}{4}, \frac{1}{2}]$, $m = 4$,
so $\frac{m}{2r} = 1$

$$\mathrm{RREhr}\left(\mathrm{P}_2; t\right) = \frac{1 + t^{\frac{1}{2}} + t^{\frac{3}{4}} + t^{\frac{5}{4}}}{(1-t)^2}$$

P_2 is $2r$-rational Gorenstein.

Theorem

Let P be a rational d-polytope with codenominator $r = \operatorname{lcm}(\mathbf{b})$, $\mathbf{0} \in \mathrm{P}$, as above. Then the following are equivalent:

1. P is r-rational Gorenstein with $(g, \mathbf{y}) \in \operatorname{hom}\left(\frac{1}{r}\mathrm{P}\right)$.
2. there exists a (necessarily unique) integer solution $(g, \mathbf{y})$ to
$$-\mathbf{a}_j\,\mathbf{y} = 1 \quad \text{for } j = 1, \dots, i$$
$$b_j\,g - r\,\mathbf{a}_j\,\mathbf{y} = b_j \quad \text{for } j = i+1, \dots, n$$
3. $\mathrm{rh}^*(\mathrm{P}; t)$ is palindromic:
$$t^{(d+1)\frac{m}{r} - \frac{k}{r}}\ \mathrm{rh}^*_m\left(\mathrm{P}; \frac{1}{t}\right) \;=\; \mathrm{rh}^*_m(\mathrm{P}; t)\,.$$
4. $(-1)^{d+1} t^{\frac{g}{r}}\,\mathrm{REhr}(\mathrm{P}; t) = \mathrm{REhr}\left(\mathrm{P}; \frac{1}{t}\right)$.
5. $\mathrm{rehr}\left(\mathrm{P}; \frac{n}{r}\right) = \mathrm{rehr}\left(\mathrm{P}; \frac{n+g}{r}\right)$ for all $n \in \mathbb{Z}_{\geq 0}$.
6. $\operatorname{hom}\left(\frac{1}{r}\mathrm{P}\right)^{\vee}$ is the cone over a lattice polytope.

Rational Gorenstein Polytopes

Theorem

Let P be a rational d-polytope with codenominator $r = \operatorname{lcm}(\mathbf{b})$, as above. Then the following are equivalent:

1. P is $2r$-rational Gorenstein with $(g, \mathbf{y}) \in \hom\left(\frac{1}{2r}\mathrm{P}\right)$.
2. there exists a (necessarily unique) integer solution $(g, \mathbf{y})$ to
$$-\mathbf{a}_j\,\mathbf{y} = 1 \quad \text{for } j = 1, \dots, i$$
$$|b_j|\,g - 2r\,\mathbf{a}_j\,\mathbf{y} = |b_j| \quad \text{for } j = i+1, \dots, n$$
3. $\operatorname{rrh}^*(\mathrm{P}; t)$ is palindromic:
$$t^{(d+1)\frac{m}{2r} - \frac{k}{2r}}\ \operatorname{rrh}^*_m\left(\mathrm{P}; \frac{1}{t}\right) \ = \ \operatorname{rrh}^*_m(\mathrm{P}; t)\,.$$
4. $(-1)^{d+1} t^{\frac{g}{2r}}\,\operatorname{RREhr}(\mathrm{P}; t) = \operatorname{RREhr}\left(\mathrm{P}; \frac{1}{t}\right)$.
5. $\operatorname{rehr}\left(\mathrm{P}; \frac{n}{2r}\right) = \operatorname{rehr}\left(\mathrm{P}; \frac{n+g}{2r}\right)$ for all $n \in \mathbb{Z}_{\geq 0}$.
6. $\hom\left(\frac{1}{2r}\mathrm{P}\right)^\vee$ is the cone over a lattice polytope.

This can be generalized to ℓr-rational Gorenstein for $\ell \in \mathbb{Z}_{>0}$.

More Examples

- $\mathrm{P}_1 := \left[-1, \frac{2}{3}\right]$ Compute: $r = 2$, $m = 6$

$$\mathrm{rh}^*_6(\mathrm{P}_1; t) = 1 + t^{\frac{1}{2}} + 2t + 3t^{\frac{3}{2}} + 4t^2 + 4t^{\frac{5}{2}} + 4t^3 + 4t^{\frac{7}{2}} + 3t^4 + 2t^{\frac{9}{2}} + t^5 + t^{\frac{11}{2}}$$

If $\mathbf{0} \in \mathrm{P}^\circ$, then $(1, 0, \ldots, 0)$ is Gorenstein point in $\hom\left(\frac{1}{r}\mathrm{P}\right)$.

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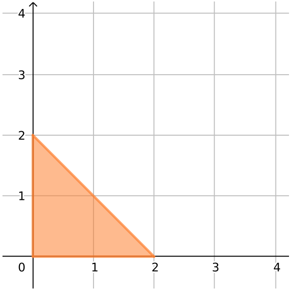
If $\mathbf{0} \in P^\circ$, then $(1, 0, \ldots, 0)$ is Gorenstein point in $\mathrm{hom}\left(\frac{1}{r}P\right)$.

- $\triangle = \mathrm{conv}\{(0,0), (2,0), (0,2)\}$ is 2-rational Gorenstein



$$\mathrm{REhr}(\triangle, t) = \frac{1 + 3t^{\frac{1}{2}} + 3t + t^{\frac{3}{2}}}{(1-t)^3}$$

$$h^*(\triangle, t) = 1 + 3t$$

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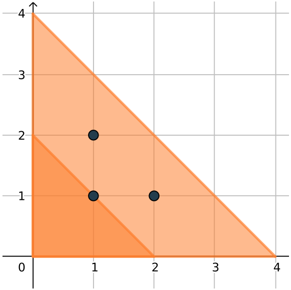

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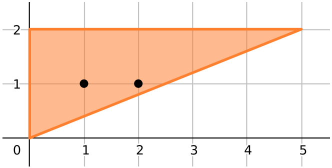
- $\triangle = \mathrm{conv}\{(0,0),(2,0),(0,2)\}$ is 2-rational Gorenstein

- $\nabla = \mathrm{conv}\{(0,0),(0,2),(5,2)\}$ is not rational Gorenstein

Thanks to Esme Bajo for suggesting this example.



$$\mathrm{rh}_2^*(\nabla; t) = 1 + 4t^{\frac{1}{2}} + 7t + 6t^{\frac{3}{2}} + 2t^2$$

Outlook

- What is a reasonable definition of "reflexive" in the rational setting?
- Connections to the Fine (1983) interior of a lattice polytope (Batyrev 2017, Batyrev–Kasprzyk–Schaller 2022)?
- Any other Ehrhart-theoretic question ...

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Thank you for your attention!

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Recall: The period p of (rational) Ehrhart quasipolynomial divides the denominator k of P.

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More examples:

1. $\Delta_3 = \text{conv}\left\{(0,0), \left(1, \frac{2}{3}\right), (3,0)\right\}$
   $\rightarrow$ **integral** period collapse but **no rational** period collapse
2. $\text{conv}\left\{(0,0,0), \left(\frac{1}{2},0,0\right), \left(0,\frac{1}{2},0\right), \left(\frac{1}{2},\frac{1}{2},0\right), \left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)\right\}$
   Fernandes, de Pina, Ramırez Alfonsın, and Robins, *On the period collapse of a family of Ehrhart quasi-polynomials*, 2021, Preprint (arXiv:2104.11025).
   $\rightarrow$ **integral** period collapse and **rational** period collapse
3. $\left[-1, \frac{2}{3}\right]$
   $\rightarrow$ **no integral** period collapse but **rational** period collapse
4. $\left[0, \frac{1}{2}\right]$
   $\rightarrow$ **no integral** period collapse, **no rational** period collapse