Promotion of Kreweras words

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Washington, Vienna

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How many ways are there . . .

for $2n$ people to cast votes for Alice and Bob, such that

- Alice and Bob both receive n votes, and
- Alice never trails Bob?

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$$\mathrm{AB}$$

$$\mathrm{AABB}, \mathrm{ABAB}$$

$$\mathrm{AAABBB}, \mathrm{AABABB}, \mathrm{AABBAB}, \mathrm{ABAABB}, \mathrm{ABABAB}$$

$$\ldots$$

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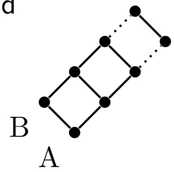

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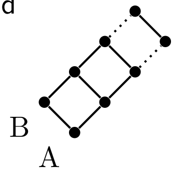

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$\ldots$

$$\frac{1}{n+1}\binom{2n}{n}$$

Promotion of Dyck words

The *promotion* $\mathrm{pr}(w)$ of a Dyck word $w = w_1 \dots w_{2n}$ is:

- remove the first letter w_1
- replace the first letter w_i such that $w_2 \dots w_i$ has more B's than A's with an A
- append B.

$$
\begin{array}{ccccccccc}
\mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{A} & \mathrm{B} & \textcircled{\mathrm{B}} & \mathrm{A} & \mathrm{B} & \\
 & & & & \downarrow & & & & \\
 & \mathrm{A} & \mathrm{B} & \mathrm{A} & \mathrm{B} & \textcircled{\mathrm{A}} & \mathrm{A} & \mathrm{B} & \mathrm{B}
\end{array}
$$

Promotion of Dyck words

The *character* of a cyclic action $\langle \mathrm{pr} \rangle \times X \to X$ of order N is the polynomial of degree less than N defined by

$$\chi_{\mathrm{pr}}(\zeta^k) = \#\text{fixed points of } \mathrm{pr}^k$$

for a primitive N-th root of unity ζ.

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Theorem (White 2007)

Promotion on Dyck words has order $2n$:

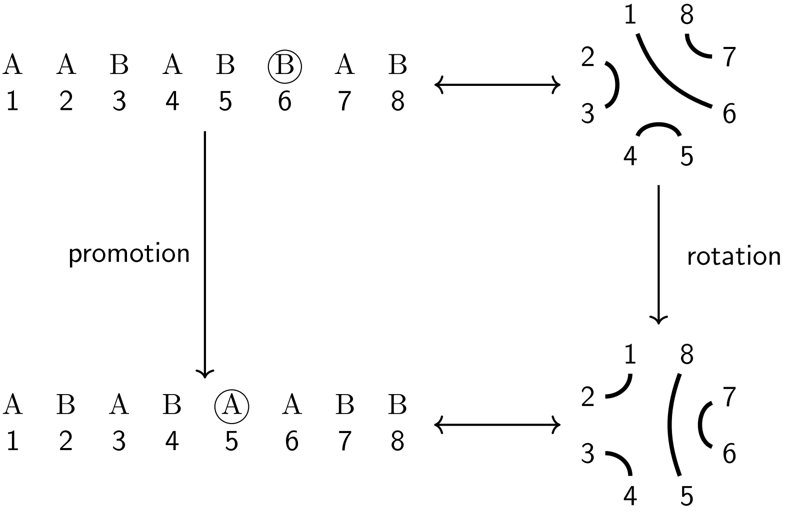
$$\mathrm{pr}^{2n}(w) = w.$$

Its character is

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \pmod{q^{2n} - 1},$$

where $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q \cdots [1]_q$ *and* $\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$.

Promotion of Dyck words is rotation of noncrossing matchings



How many ways are there . . .

for $3n$ people to cast votes for Alice, Bob and Charlie, such that

- Alice, Bob and Charlie all receive n votes, and
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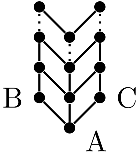

ABC, ACB

AABBCC,AABCBC,AABCCB,ABABCC,ABACBC,ABACCB,ABCABC,ABCACB,...

. . .

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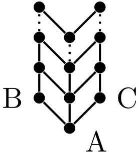

ABC, ACB

AABBCC,AABCBC,AABCCB,ABABCC,ABACBC,ABACCB,ABCABC,ABCACB,...

. . .

Theorem (Kreweras 1965)

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n}$$

Promotion of Kreweras words

The *promotion* $\mathrm{pr}(w)$ of a Kreweras word $w = w_1 \dots w_{3n}$ is:

- remove the first letter w_1
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- append w_i.

$$\begin{array}{ccccccccccc} \mathrm{A} & \mathrm{A} & \mathrm{B} & \textcircled{\mathrm{B}} & \mathrm{C} & \mathrm{A} & \mathrm{C} & \mathrm{C} & \mathrm{B} & \\ & & & & \downarrow & & & & & \\ & \mathrm{A} & \mathrm{B} & \textcircled{\mathrm{A}} & \mathrm{C} & \mathrm{A} & \mathrm{C} & \mathrm{C} & \mathrm{B} & \mathrm{B} \end{array}$$

Promotion of Kreweras words

Theorem (NEW)

Promotion on Kreweras words has order $6n$*:* $\mathrm{pr}^{3n}(w)$ *is obtained from* w *by switching all* B*'s and* C*'s.*

Conjecture (NEW)

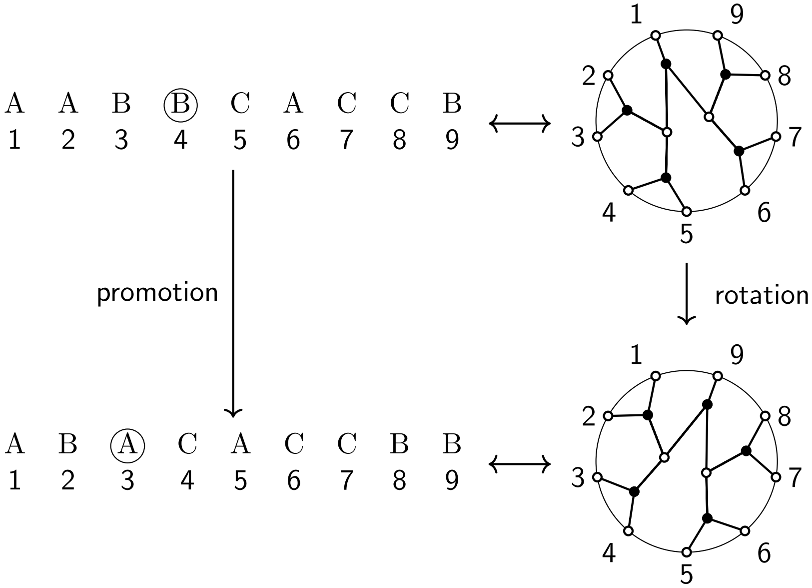
Its character is

$$\frac{[2]_q^{2n}[3n]_{q^2}!}{[n+1]_{q^2}![2n+1]_q!} \pmod{q^{6n}-1}.$$

Remark:

$$\frac{4^n}{(n+1)(2n+1)}\binom{3n}{n} = \frac{2^{2n}(3n)!}{(n+1)!(2n+1)!}$$

Promotion of Kreweras words is rotation of webs

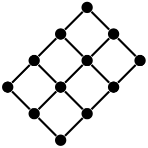


Background: posets of size N with $\mathrm{pr}^{2N} = 1$

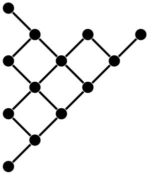
Promotion generalizes to linear extensions of arbitrary posets

Theorem (Haiman, Haiman and Kim 1992)

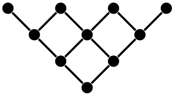
Promotion on a shape or a shifted shape has order N or $2N$ if and only if the shape is one of the following:



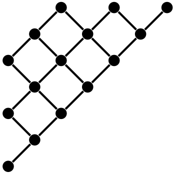
rectangle



shifted double staircase



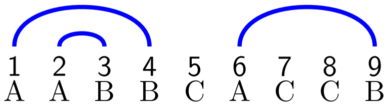
staircase

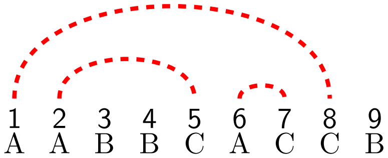


shifted trapezoid

Part 1: The order of promotion

blue and crimson noncrossing matchings:





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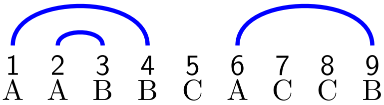
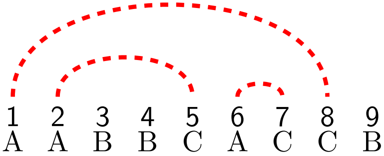
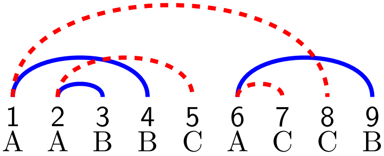


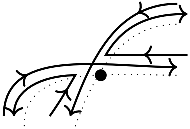


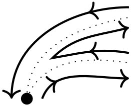
Kreweras bump diagram:



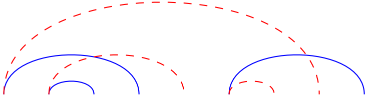
Part 1: The order of promotion

rules of the road:





trip permutation:



	1	2	3	4	5	6	7	8	9
w	A	A	B	B	C	A	C	C	B
σ_w									

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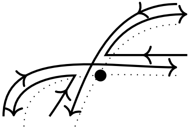

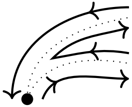

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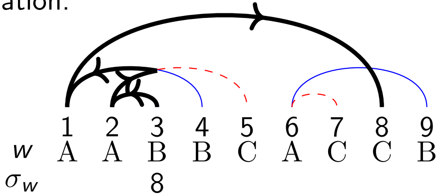

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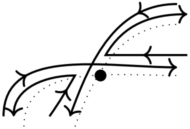

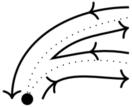

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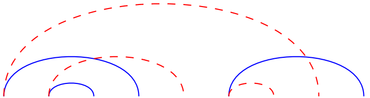

	1	2	3	4	5	6	7	8	9
w	A	A	B	B	C	A	C	C	B
σ_w	4	3	8	5	2	7	1	9	6

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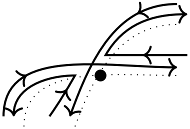

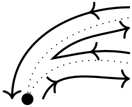

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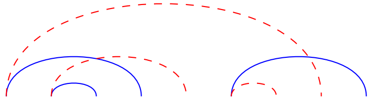
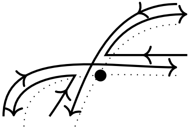
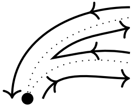


	1	2	3	4	5	6	7	8	9	
w	A	A	B	B	C	A	C	C	B	
σ_w	4	3	8	5	2	7	1	9	6	
ε_w	B	B	C	C	B	C	B	B	C	(next non-A following σ_w)

Part 1: The order of promotion

rules of the road:





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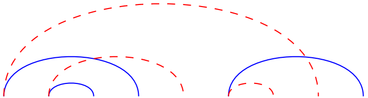


	1	2	3	4	5	6	7	8	9	
w	A	A	B	B	C	A	C	C	B	
σ_w	4	3	8	5	2	7	1	9	6	
ε_w	B	B	C	C	B	C	B	B	C	(next non-A following σ_w)

Lemma

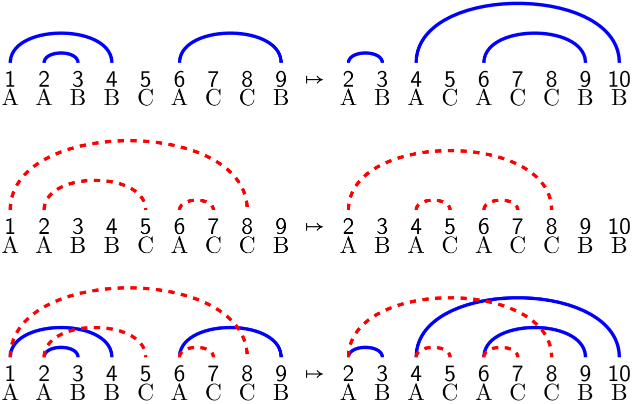
σ_w and ε_w determine w.

Part 1: The order of promotion

Lemma

$$\sigma_{\mathrm{pr}(w)} = \mathrm{rot}(\sigma_w)$$

$$\varepsilon_{\mathrm{pr}(w)} = [\varepsilon_w(2), \varepsilon_w(3), \ldots, \varepsilon_w(3n), -\varepsilon_w(1)]$$



Corollary

$\mathrm{pr}^{3n}(w)$ *is obtained from w be swapping B's and C's.*

Background: Kuperberg's $\mathfrak{sl}_3$-webs (1996)

An $\mathfrak{sl}_3$-*web* is a planar graph $\mathcal{W}$, embedded in a disk, with $3n$ white *boundary vertices* arranged counterclockwise, such that

- $\mathcal{W}$ is *trivalent*, i.e., boundary vertices have degree 1, all others have degree 3,
- $\mathcal{W}$ is *bipartite*, i.e., properly coloured black and white, and
- $\mathcal{W}$ is *irreducible*, i.e., no internal faces with fewer than 6 sides.

$\mathfrak{sl}_3$-webs are in bijection with $3 \times n$ standard tableaux.

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Theorem (NEW)

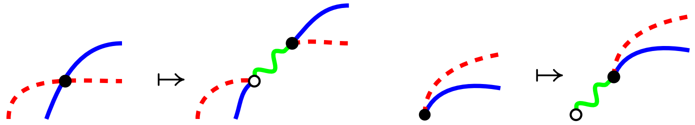
There is a bijection between Kreweras words and Kreweras webs, $\mathfrak{sl}_3$-webs

- *without internal faces having a multiple of 4 sides, and*
- *where each connected component is assigned one of two colours (blue or crimson),*

which intertwines promotion and rotation.

Part 2: Kreweras words to Kreweras webs

breaking apart crossings:



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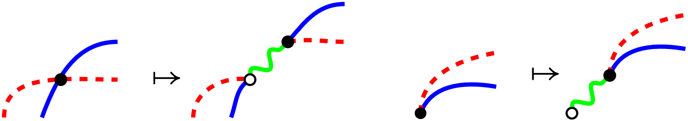

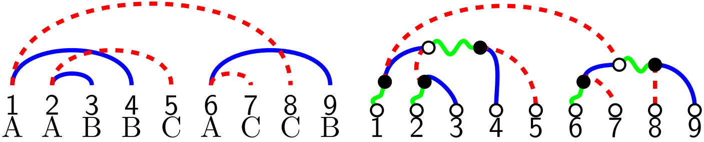

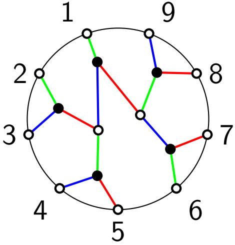

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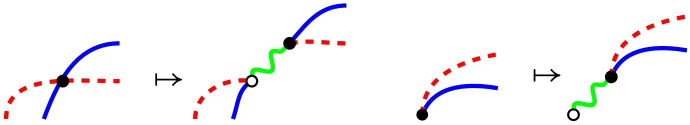
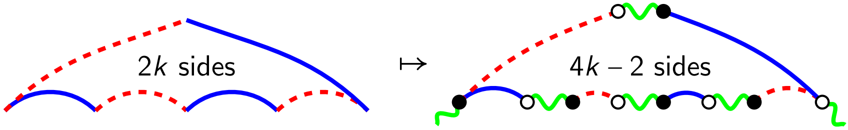


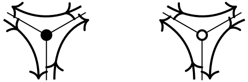
internal faces have $4k-2$ sides, with $k \geq 2$:



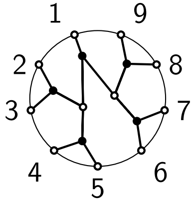
(converse non-trivial)

Part 2: Trips in Kreweras webs

rules of the road:



trip permutation:



$\mapsto$

1	2	3	4	5	6	7	8	9
4	3	8	5	2	7	1	9	6

Theorem

The trip permutation of the Kreweras bump diagram is the trip permutation of the Kreweras web.

Theorem (Postnikov 2006)

The trip permutation determines the Kreweras web.

Corollary

Promotion of a Kreweras word is rotation of the Kreweras web.