Soliton cellular automata for the affine general linear Lie superalgebra

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Joint work with Mitchell Ryan

The University of Queensland

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Outline

What is a Box-Ball System?

- The *box-ball system (BBS)* is an ultradiscrete (discrete in time and space) analogue of the *Korteweg–de Vries (KdV) equation*,

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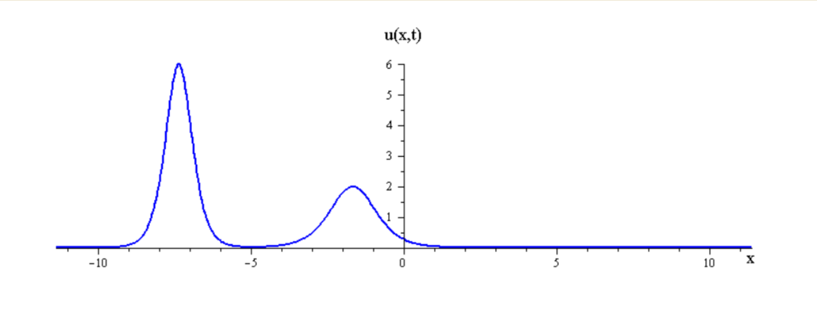
- The KdV equation admits *soliton* solutions, which are solitary waves moving through the channel.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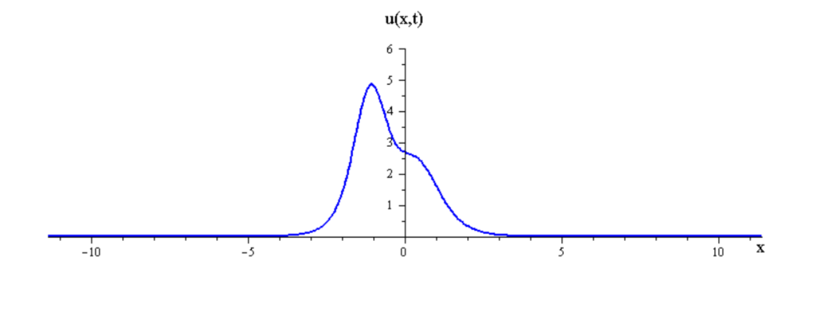
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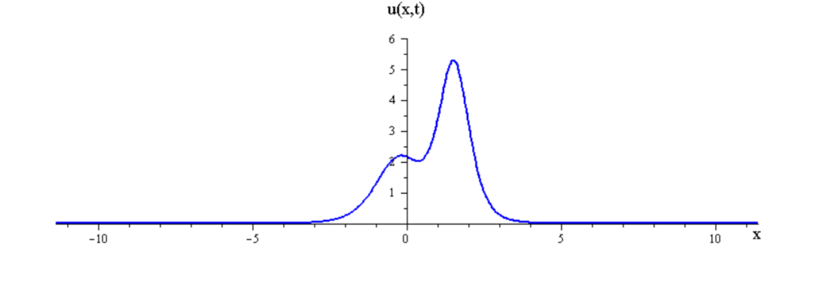
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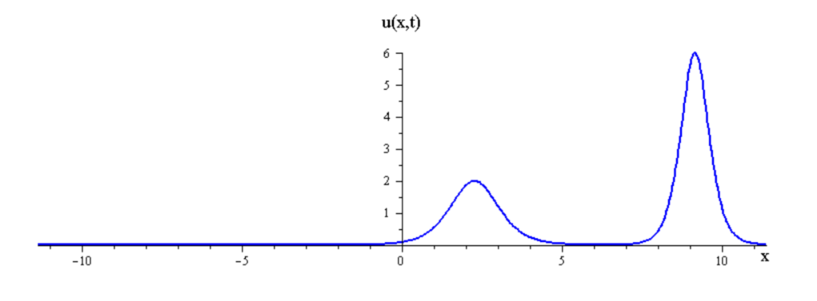
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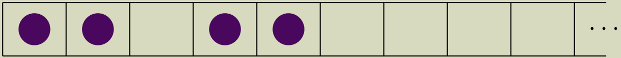
Example

$t = 0$



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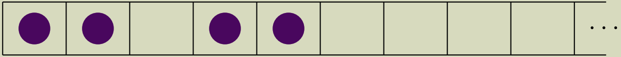

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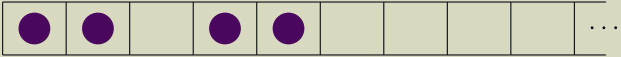

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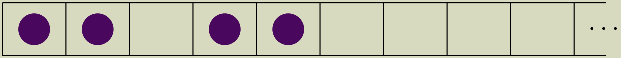

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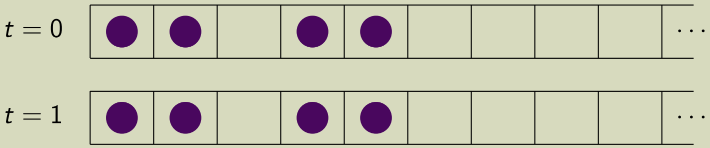

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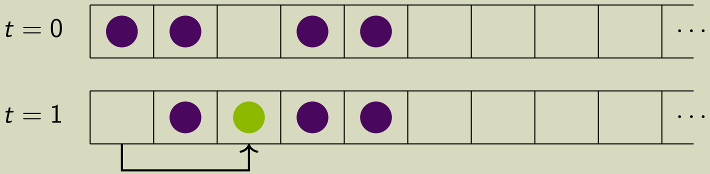

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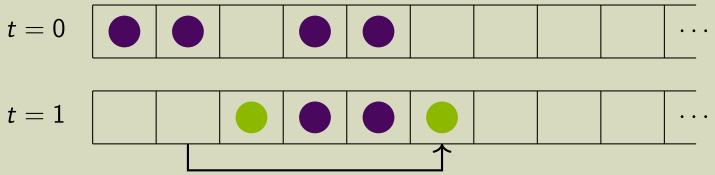

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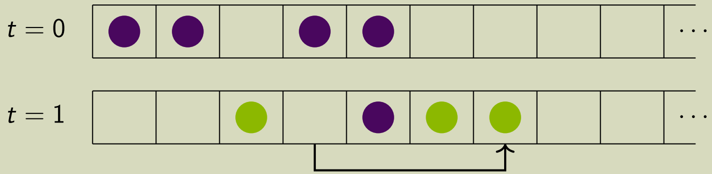

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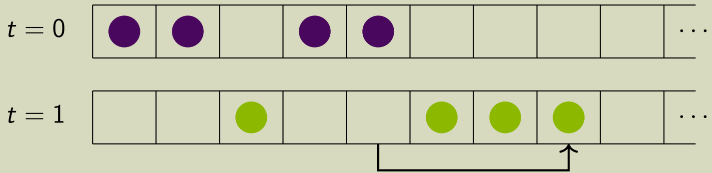

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These conditions are analogous to the defining properties of a soliton solution to the KdV equation.

BBS Example

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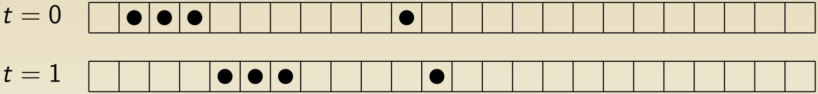

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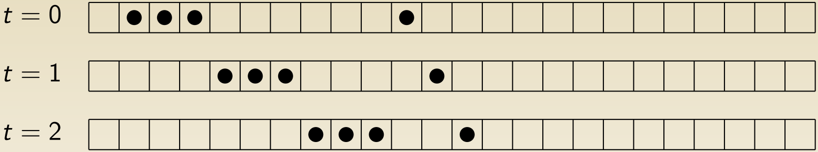

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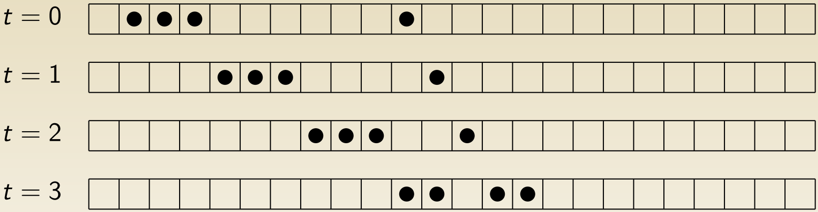

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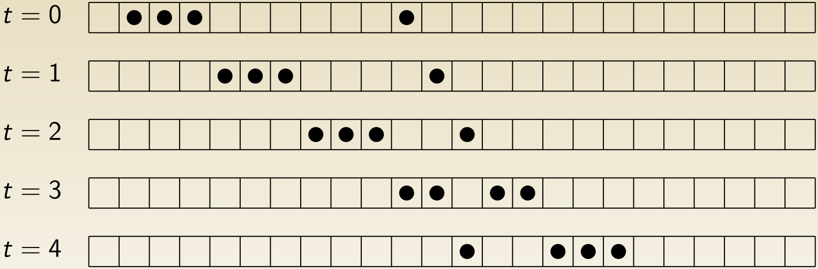

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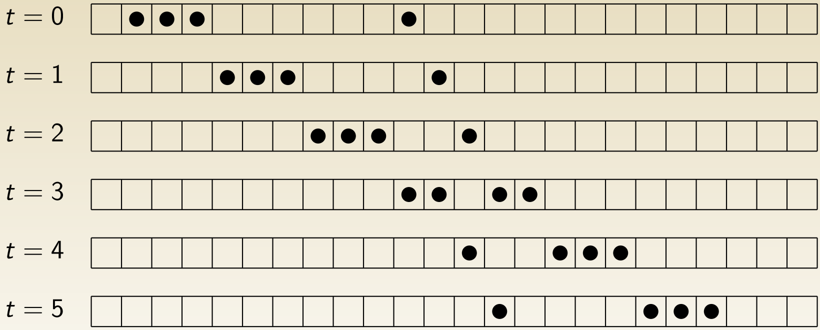

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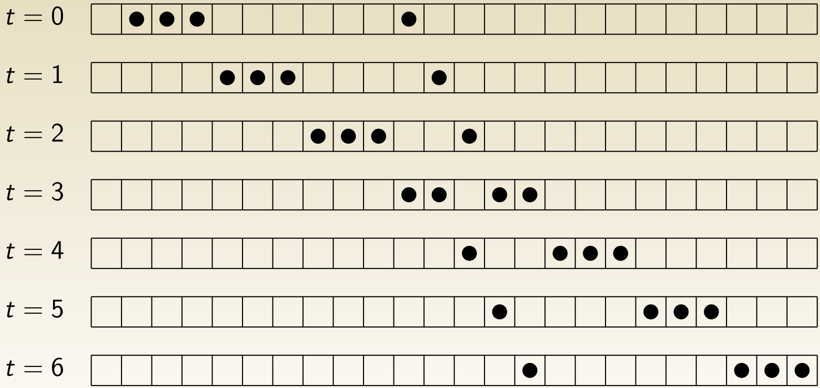

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is a standard basis for $\mathfrak{gl}_2$, then

$$\begin{tikzcd} v_1 \arrow[loop left, "h_1"] \arrow[rr, bend left, "f_1"] & & v_2 \arrow[loop right, "h_1"] \arrow[ll, bend left, "e_1"] \end{tikzcd}$$

(up to scalar multiples).

Crystal Example

We can encode these relationships with a *crystal*,

$$\begin{tikzcd} v_1 \arrow[loop left, "h_1"] \arrow[r, bend left, "f_1"] & v_2 \arrow[loop right, "h_1"] \arrow[l, bend left, "e_1"] \end{tikzcd}$$

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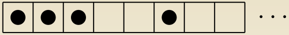
- Tensor products of these crystals are connected.

Encoding a BBS with crystals

We can encode a BBS state, such as



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where

- $\boxed{1}$ represents an empty box (aka a *vacuum element*)
- $\boxed{2}$ represents a full box.

Carrier and R-matrix

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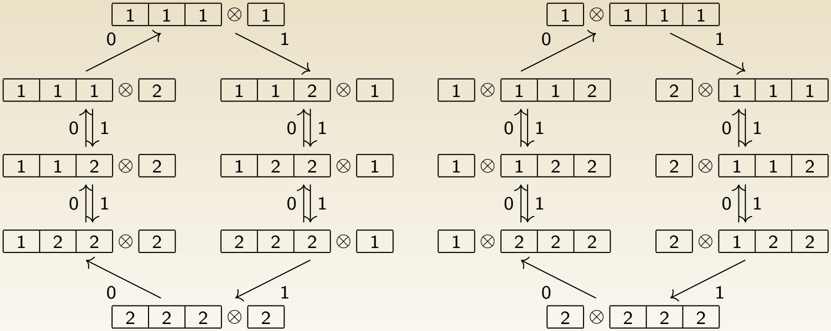
- We can move the carrier using the *combinatorial R-matrix*:

$$R\colon B(\boxed{}\boxed{}\boxed{}) \otimes B(\boxed{}) \longrightarrow B(\boxed{}) \otimes B(\boxed{}\boxed{}\boxed{})$$

Carrier and R-matrix

If we swap the order of the tensor products, we obtain isomorphic crystals:



The R-matrix is this isomorphism.

Time evolution using the R-matrix

$$\boxed{\bullet}\boxed{\bullet}\boxed{\bullet}\boxed{}\boxed{}\boxed{\bullet}\boxed{}\boxed{}\ \cdots$$

$$t = 0 \qquad \boxed{2} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \cdots$$

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Time evolution

$$\boxed{1\,|\,1\,|\,1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \cdots$$

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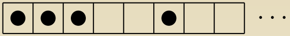

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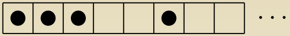

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Time evolution

$$\boxed{1} \otimes \boxed{1} \otimes \boxed{1\;2\;2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \cdots$$

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Time evolution

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Time evolution using the R-matrix

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \bullet & \bullet & \bullet & & & \bullet & & \\\hline\end{array}\ \cdots$$

$$t=0 \qquad \boxed{2}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{1}\otimes\cdots$$

Time evolution

$$\boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\begin{array}{|c|c|c|}\hline 1 & 1 & 1\\\hline\end{array}\otimes\cdots$$

$$t=1 \qquad \boxed{1}\otimes\boxed{1}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{2}\otimes\boxed{1}\otimes\cdots$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & & & \bullet & \bullet & & \bullet & \bullet & \\\hline\end{array}\ \cdots$$

Generalisations

- Different ‘coloured’ balls for $\widehat{\mathfrak{gl}}_n$ crystals for all $n \geq 2$ (Hatayama–Kuniba–Takagi, 2000).

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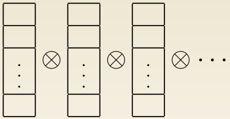

Generalisations

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$$\begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline\end{array} \otimes \cdots$$

- Single-box tableaux from crystals of supersymmetric Lie algebras (Hikami–Inoue, 2000).

Outline

General Linear Lie Superalgebra

Definition

The *general linear Lie superalgebra* $\mathfrak{gl}(m|n)$ is the set of all linear transformations of the $(m|n)$-dimensional *super vector space*, equipped with the *super-commutator* $[X, Y] = XY - (-1)^{|X||Y|}YX$.

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- Using the structure of this superalgebra we describe the generalised *supersymmetric box-ball system* (SBBS).

Crystals in $U_q(\mathfrak{gl}(m|n))$

- Let $v_{\overline{m}}, v_{\overline{m-1}}, \ldots, v_{\overline{1}}, v_1, \ldots, v_{n-1}, v_n$ be the standard basis vectors of the $(m|n)$-dimensional super vector space over which $\mathfrak{gl}(m|n)$ acts.

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- The crystal for $\mathfrak{gl}(m|n)$ is

$$\boxed{\overline{m}} \xrightarrow{\overline{m-1}} \boxed{\overline{m-1}} \xrightarrow{\overline{m-2}} \cdots \xrightarrow{\overline{1}} \boxed{\overline{1}} \xrightarrow{0} \boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n}$$

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- We call the barred values the *bosonic* entries, and the unbarred *fermionic* entries.

A tableau of shape $Y^{r,1}$ is a column of values as follows

$$r \left\{ \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_r \\ \hline \end{array} \right.$$

where $x_i \in \{\overline{m}, \overline{m-1}, \ldots, \overline{1}, 1, \ldots, n-1, n\}$.

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An empty box, or *vacuum*, is the tableau

$$r\left\{\begin{array}{|c|}\hline \overline{m} \\ \hline \overline{m-1} \\ \hline \vdots \\ \hline \overline{m-r+1} \\ \hline \end{array}\right.$$

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- From Kwon–Okado, we have a $U_q(\widehat{\mathfrak{gl}}(m|n))$-crystal structure and a combinatorial R-matrix that we can use to define time evolution of the system.

The R-matrix and Schensted's Bumping Algorithm

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$$\mathrm{col}\left(\begin{array}{|c|c|c|}\hline \boldsymbol{\overline{4}} & \overline{4} & 1 \\ \hline \boldsymbol{\overline{3}} & \overline{3} & 2 \\ \hline \mathbf{1} & 2 & 3 \\ \hline \end{array}\right) = 1\,2\,3\,\overline{4}\,\overline{3}\,2\,\boldsymbol{\overline{4}}\,\boldsymbol{\overline{3}}\,\mathbf{1}$$

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$$\overline{2} \rightarrow \begin{array}{|c|c|c|c|} \hline \overline{3} & \overline{3} & 1 & 3 \\ \hline \overline{2} & 1 & 2 & 5 \\ \hline \end{array}$$

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Schensted's Bumping Algorithm

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$$\begin{array}{cccc} & & & 2 \\ & & & \downarrow \\ \hline \multicolumn{1}{|c|}{\bar{3}} & \multicolumn{1}{c|}{\bar{3}} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\mathbf{3}} \\ \hline \multicolumn{1}{|c|}{\bar{2}} & \multicolumn{1}{c|}{\bar{2}} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

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$$\begin{array}{ccccc} & & & & 3 \\ & & & & \downarrow \\ \end{array}$$

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Schensted's Bumping Algorithm

$$\begin{array}{|c|c|c|c|c|} \hline \overline{3} & \overline{3} & 1 & 2 & 3 \\ \hline \overline{2} & \overline{2} & 1 & 5 & \multicolumn{1}{c}{} \\ \cline{1-4} \end{array}$$

R-matrix and Schensted's Bumping Algorithm

[Kwon–Okado 2021] Suppose we have the tensor product of tableaux $x \otimes y$. Then the combinatorial R-matrix sends $x \otimes y$ to $\tilde{y} \otimes \tilde{x}$, if and only if $\mathrm{col}\,(y) \to x = \mathrm{col}(\tilde{x}) \to \tilde{y}$.

Computation of the R-matrix

Set $x = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$, $y = \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$, $\tilde{y} = \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$, $\tilde{x} = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & 1 \\ \hline \overline{3} & \overline{3} & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$.

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Insertion

$$\mathrm{col}(y) \to x = \bar{3}12 \to \begin{array}{|c|c|c|}\hline \bar{4} & \bar{4} & \bar{3} \\ \hline \bar{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

Computation of the R-matrix

Set $\quad x = \begin{array}{|c|c|c|}\hline \bar{4} & \bar{4} & \bar{3} \\ \hline \bar{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad y = \begin{array}{|c|}\hline \bar{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}, \quad \tilde{y} = \begin{array}{|c|}\hline \bar{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}, \quad \tilde{x} = \begin{array}{|c|c|c|}\hline \bar{4} & \bar{4} & 1 \\ \hline \bar{3} & \bar{3} & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$

Insertion

$$\mathrm{col}(y) \to x = 12 \to \begin{array}{|c|c|c|c|}\hline \bar{4} & \bar{4} & \bar{3} & 3 \\ \hline \bar{3} & \bar{3} & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

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Insertion

$$\mathrm{col}(y) \to x = 2 \to \begin{array}{|c|c|c|c|}\hline \bar{4} & \bar{4} & \bar{3} & 3 \\ \hline \bar{3} & \bar{3} & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad .$$

Computation of the R-matrix

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Insertion

$$\mathrm{col}(y) \to x = \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} & 3 \\ \hline \overline{3} & \overline{3} & 1 & \\ \cline{1-3} 1 & 2 & 3 & \\ \cline{1-3} 1 & & & \\ \cline{1-1} 2 & & & \\ \cline{1-1} \end{array}$$

Computation of the R-matrix

Set $x = \begin{array}{|c|c|c|}\hline \bar{4} & \bar{4} & \bar{3} \\ \hline \bar{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \quad y = \begin{array}{|c|}\hline \bar{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}, \quad \tilde{y} = \begin{array}{|c|}\hline \bar{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}, \quad \tilde{x} = \begin{array}{|c|c|c|}\hline \bar{4} & \bar{4} & 1 \\ \hline \bar{3} & \bar{3} & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$

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Similarly, $\mathrm{col}(\tilde{x}) \to \tilde{y} = 1\,2\,3\,\bar{4}\,\bar{3}\,2\,\bar{4}\,\bar{3}\,1 \to \begin{array}{|c|}\hline \bar{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$

Computation of the R-matrix

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Computation of the R-matrix

Thus,

$$R\left(\begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & \overline{3} \\ \hline \overline{3} & 1 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 2 \\ \hline \end{array}\right) = \begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline \overline{4} & \overline{4} & 1 \\ \hline \overline{3} & \overline{3} & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\cdot$ represent the vacuum element.

$$t = 0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \begin{smallmatrix}\bar{1}\\2\end{smallmatrix}\ \begin{smallmatrix}\bar{2}\\1\end{smallmatrix}\ \cdot\ \cdot\ \begin{smallmatrix}\bar{1}\\1\end{smallmatrix}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

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$$t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \substack{\bar{1}\\2}\ \substack{\bar{2}\\1}\ \cdot\ \substack{\bar{1}\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\bullet$ represent the vacuum element.

$$t=0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=2 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\cdot$ represent the vacuum element.

$$t=0 \quad \begin{array}{ccccccccccccccccccc} & & & & \bar{1} & \bar{2} & & & \bar{1} & & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & 2 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$t=1 \quad \begin{array}{ccccccccccccccccccc} & & & & & & \bar{1} & \bar{2} & & \bar{1} & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$t=2 \quad \begin{array}{ccccccccccccccccccc} & & & & & & & & \bar{1} & \bar{2} & \bar{1} & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$t=3 \quad \begin{array}{ccccccccccccccccccc} & & & & & & & & & & \bar{1} & 1 & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\bullet$ represent the vacuum element.

$$
\begin{array}{l}
t=0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \substack{\bar{1}\\2}\ \substack{\bar{2}\\1}\ \cdot\ \cdot\ \substack{\bar{1}\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2em]
t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \substack{\bar{1}\\2}\ \substack{\bar{2}\\1}\ \cdot\ \substack{\bar{1}\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2em]
t=2 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \substack{\bar{1}\\2}\ \substack{\bar{2}\\1}\ \substack{\bar{1}\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2em]
t=3 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \substack{\bar{1}\\2}\ \substack{1\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\[2em]
t=4 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \substack{1\\2}\ \substack{\bar{1}\\1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\cdot$ represent the vacuum element.

$$t=0 \quad \cdot\;\cdot\;\cdot\;\cdot\;\substack{\bar{1}\\2}\;\substack{\bar{2}\\1}\;\cdot\;\cdot\;\substack{\bar{1}\\1}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$t=1 \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\substack{\bar{1}\\2}\;\substack{\bar{2}\\1}\;\cdot\;\substack{\bar{1}\\1}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$t=2 \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\substack{\bar{1}\\2}\;\substack{\bar{2}\\1}\;\substack{\bar{1}\\1}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$t=3 \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\substack{\bar{1}\\2}\;\substack{1\\1}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$t=4 \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\substack{1\\2}\;\substack{\bar{1}\\1}\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$t=5 \quad \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\substack{\bar{2}\\1}\;\substack{\bar{1}\\2}\;\substack{\bar{1}\\1}\;\cdot\;\cdot\;\cdot$$

Example $\widehat{\mathfrak{gl}}(2|2)$

Consider $\widehat{\mathfrak{gl}}(2|2)$. Let $\cdot$ represent the vacuum element.

$$t=0 \quad \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=1 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=2 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=3 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{1}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=4 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{1}{2}\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$t=5 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot\ \cdot\ \cdot$$

$$t=6 \quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \genfrac{}{}{0pt}{}{\bar{2}}{1}\ \cdot\ \genfrac{}{}{0pt}{}{\bar{1}}{2}\ \genfrac{}{}{0pt}{}{\bar{1}}{1}\ \cdot$$

Soliton Speed

Let $S \in B(Y^{r,1})^{\otimes s}$ be given by

$$\begin{array}{|c|}\hline x_{11} \\ \hline x_{21} \\ \hline \vdots \\ \hline x_{r1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline x_{12} \\ \hline x_{22} \\ \hline \vdots \\ \hline x_{r2} \\ \hline \end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline x_{1s} \\ \hline x_{2s} \\ \hline \vdots \\ \hline x_{rs} \\ \hline \end{array}$$

Soliton Speed

Let $S \in B(Y^{r,1})^{\otimes s}$ be given by

increasing (strict if bosonic)

$$\begin{array}{|c|}\hline x_{11} \\ \hline x_{21} \\ \hline \vdots \\ \hline x_{r1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline x_{12} \\ \hline x_{22} \\ \hline \vdots \\ \hline x_{r2} \\ \hline \end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline x_{1s} \\ \hline x_{2s} \\ \hline \vdots \\ \hline x_{rs} \\ \hline \end{array}$$

Soliton Speed

Let $S \in B(Y^{r,1})^{\otimes s}$ be given by

$$\xrightarrow{\text{decreasing (strict if fermionic)}}$$

$$\begin{array}{c} \text{increasing} \\ \text{(strict if} \\ \text{bosonic)} \end{array} \Bigg\downarrow \quad \begin{array}{|c|}\hline x_{11} \\ \hline x_{21} \\ \hline \vdots \\ \hline x_{r1} \\ \hline \end{array} \otimes \begin{array}{|c|}\hline x_{12} \\ \hline x_{22} \\ \hline \vdots \\ \hline x_{r2} \\ \hline \end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline x_{1s} \\ \hline x_{2s} \\ \hline \vdots \\ \hline x_{rs} \\ \hline \end{array}$$

Soliton Speed

Let $S \in B(Y^{r,1})^{\otimes s}$ be given by

decreasing (strict if fermionic) $\longrightarrow$

increasing (strict if bosonic) $\downarrow$

$$
\begin{array}{|c|}\hline x_{11} \\ \hline x_{21} \\ \hline \vdots \\ \hline x_{r1} \\ \hline \end{array}
\otimes
\begin{array}{|c|}\hline x_{12} \\ \hline x_{22} \\ \hline \vdots \\ \hline x_{r2} \\ \hline \end{array}
\otimes \cdots \otimes
\begin{array}{|c|}\hline x_{1s} \\ \hline x_{2s} \\ \hline \vdots \\ \hline x_{rs} \\ \hline \end{array}
\qquad
\begin{array}{l} \uparrow < \overline{m-r} \\ \downarrow \geq \overline{m-r} \end{array}
$$

Soliton Speed

Let $S \in B(Y^{r,1})^{\otimes s}$ be given by

decreasing (strict if fermionic) $\longrightarrow$

$$\text{increasing (strict if bosonic)} \Big\downarrow \quad \begin{array}{|c|}\hline x_{11}\\ \hline x_{21}\\ \hline \vdots\\ \hline x_{r1}\\ \hline\end{array} \otimes \begin{array}{|c|}\hline x_{12}\\ \hline x_{22}\\ \hline \vdots\\ \hline x_{r2}\\ \hline\end{array} \otimes \cdots \otimes \begin{array}{|c|}\hline x_{1s}\\ \hline x_{2s}\\ \hline \vdots\\ \hline x_{rs}\\ \hline\end{array} \quad \begin{array}{l} \uparrow < \overline{m-r} \\ \\ \downarrow \geq \overline{m-r} \end{array}$$

Then S moves with speed equal to its length.

Soliton Collisions

Theorem (Ryan–S. 2022+; Soliton Collisions)

If S and T both have the form on the previous slide and we draw the line directly above the last row, then S and T are stable under collision.

Soliton Collisions

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Example

Consider height-3 tableaux for $\widehat{\mathfrak{gl}}(4|2)$ (so $m = 4$, $r = 3$).

$$\begin{array}{|c|}\hline \overline{3}\\\hline \overline{2}\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline \overline{4}\\\hline \overline{2}\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline \overline{4}\\\hline \overline{3}\\\hline \overline{1}\\\hline\end{array} \qquad \begin{array}{l} \uparrow < \overline{m-r} = \overline{1} \\[2em] \downarrow \geq \overline{m-r} = \overline{1} \end{array}$$

satisfies the assumptions of the theorem.

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does NOT satisfy the assumptions of the theorem.

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Bigger Picture?

- There is a super analog of the KdV equation. What is the relationship with our system?

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- To the supersymmetric Toda lattice?

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