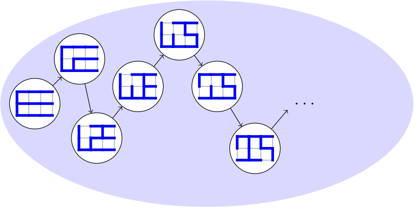
Log-concavity in matroids and expanders



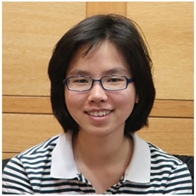
$\leftrightarrow$

Cynthia Vinzant

based on joint works with

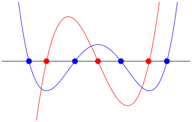
Nima Anari, Kuikui Liu, Shayan Oveis Gharan & Thuy-Duong Vuong



Warm up: real rooted polynomials

A univariate polynomial $f \in \mathbb{R}[x]$ is real rooted if all of its zeros (over $\mathbb{C}$) are real.



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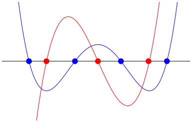

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Discrete log-concavity: If $f = \sum_{k=0}^{n} a_k x^k$ is real rooted and has nonnegative coefficients, then $(a_0, \ldots, a_n)$ is *ultra log-concave*:

$$\frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} \leq \left(\frac{a_k}{\binom{n}{k}}\right)^2 .$$

(Newton's inequalities)

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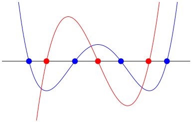

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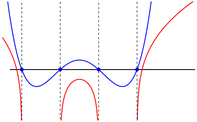
Continuous log-concavity: If $f = \sum_{k=0}^{n} a_k x^k$ is real rooted and has nonnegative coefficients, f is *log-concave on* $\mathbb{R}_+$:

$$f = \prod_{i=1}^{n}(x - \lambda_i) \;\Rightarrow\; \log(f)'' = \sum_{i=1}^{n} \frac{-1}{(x-\lambda_i)^2} \leq 0$$



Multivariate generalization: real stability

$f \in \mathbb{R}[x_1, \ldots, x_n]$ is stable if f has no zeros in $\mathcal{H}_+^n = \{z \in \mathbb{C}^n : \mathrm{Im}(z) \in \mathbb{R}^n_{>0}\}$.

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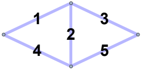
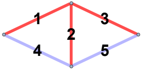
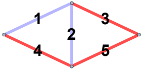
Example: $f = \sum_{T \in \mathcal{T}} \prod_{e \in T} x_e$ where $\mathcal{T} = \{$spanning trees of $G\}$

$\Big\{$ spanning trees of 

$\Big\} = \Big\{$ 

$, \ldots,$ 

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$$\begin{aligned} &x_1x_2x_3 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 \\ &+x_1x_4x_5 + x_2x_3x_4 + x_2x_4x_5 + x_3x_4x_5 \end{aligned} = \det\begin{pmatrix} x_1 + x_4 & -x_4 & 0 \\ -x_4 & x_2 + x_4 + x_5 & -x_5 \\ 0 & -x_5 & x_3 + x_5 \end{pmatrix}$$

Stable Polynomials in Combinatorics and Optimization

Convex Optimization (Hyperbolicity and Interior Point Methods)

Güler (1997), Truong, Tuncel (2004), Renegar (2006)

See also: Hyperbolic Polynomials and Convex Analysis
by Bauschke, Güler, Lewis, Sendov (2001)

Operator theory and Ramanujan graphs (Interlacing families)

Marcus, Spielman, Srivastava (2013)

Counting, Sampling, Negative dependence

Gurvits (2008), Anari, Oveis Gharan, Rezaei (2016),
Li, Jegelka, Sra (2016), Straszak, Vishnoi (2017).

See also: Negative dependence and the geometry of polynomials
by Borcea, Brändén, Liggett (2009)

Stable polynomials and negative dependence

Theorem (Brändén 2007) If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is stable, then for every $i, j \in [n]$, the polynomial

$$\Delta_{ij}(f) = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}$$

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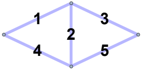
Example: T = spanning tree of 

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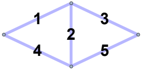
Example: T = spanning tree of 

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Choe, Oxley, Sokal, Wagner (2002): close connection with matroids

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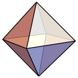
Independence complex: $\mathcal{I} = \{S \subseteq [n] : S \in B \text{ for some } B \in \mathcal{B}\}$ 

Basis Polytope: $\text{conv}\{\mathbf{1}_B : B \in \mathcal{B}\} \subset [0,1]^n$ 

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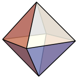

Non-ex: $\{\{1,2\},\{3,4\}\}$ not the set of bases of a matroid

Matroids and combinatorial Hodge theory

In 2015, Adiprasito, Huh, and Katz develop combinatorial Hodge theory and use it to show the log-concavity of the sequence $i_0, i_1, \ldots i_n$ where $i_k = \#\{I \in \mathcal{I} : |I| = k\}$ for any matroid $([n], \mathcal{I})$.

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Common theme: signatures of quadratic forms on subspaces

Log-concave polynomials

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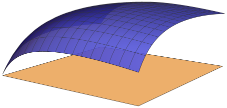

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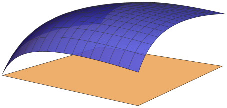

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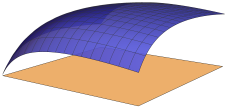

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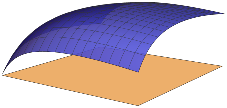

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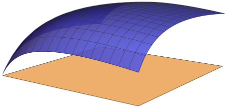

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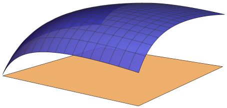

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Brändén, Huh (2019): develop equivalent Lorentzian polynomials and show connection with matroids, M-convex functions

Log-concavity and the second eigenvalue

Remark. If f is log-concave at a point $a \in \mathbb{R}_+^n$ then

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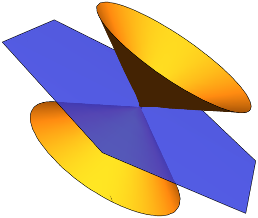

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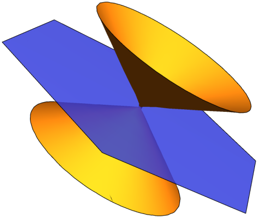

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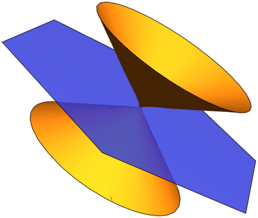

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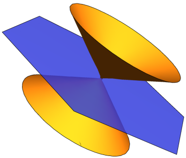
$$\nabla^2 f = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \mathbf{1}\mathbf{1}^T - \mathrm{Id}_4$$ (one pos. eig. val.)

Implications for discrete log-concavity



If f is homogeneous of degree ≥ 2 with nonnegative coefficients, then

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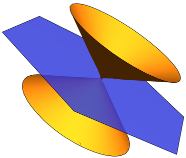

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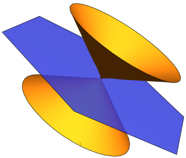

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Why? Take $q = (\frac{\partial}{\partial x})^{k-1}(\frac{\partial}{\partial y})^{n-k-1} f$. Then

$$\det\left(\nabla^2 q\right) = (n!)^2 \left(\frac{a_{k-1}}{\binom{n}{k-1}} \cdot \frac{a_{k+1}}{\binom{n}{k+1}} - \frac{a_k^2}{\binom{n}{k}^2} \right) \leq 0.$$

Log-concavity for matroid polynomials

Theorem. If $f = \sum_{S \in \binom{[n]}{d}} c_S \mathbf{x}^S$ is strongly log-concave then $\{S : c_S \neq 0\}$ are the bases of a matroid. Moreover, for any matroid with bases $\mathcal{B}$ and independent sets $\mathcal{I}$

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Non-example. $f = x_1x_2 + x_3x_4$

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Corollary: For any matroid $M = ([n], \mathcal{I})$, the sequence $(i_0, \ldots, i_n)$ with $i_k = \#\{I : I \in \mathcal{I}, |I| = k\}$ is ultra log-concave.

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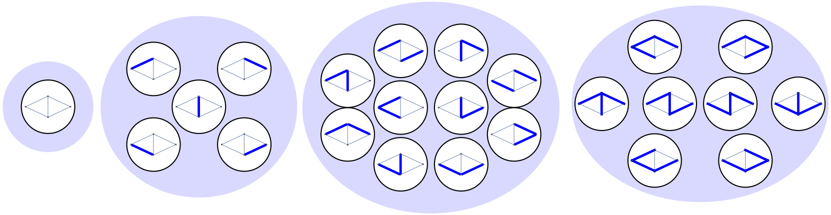

$i_0 = 1$ $\quad$ $i_1 = 5$ $\quad$ $i_2 = 10$ $\quad$ $i_3 = 8$

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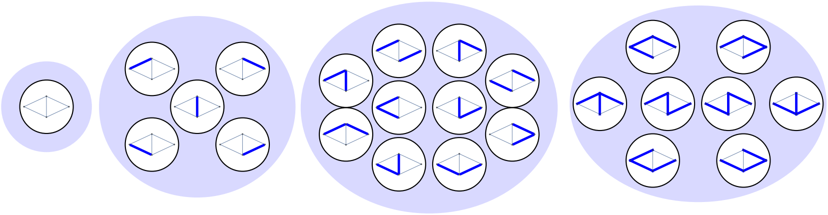

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$y^5 + 5xy^4 + 10x^2y^3 + 8x^3y^3$ is strongly log-concave on $\mathbb{R}_+^2$

A local to global theorem for log-concavity

Call f indecomposable if the graph $([n], \{\{i, j\} : \frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0\})$ is connected

e.g. $x_1x_2 + x_2x_3 + x_3x_4$ is indecomposable, $x_1x_2 + x_3x_4$ is not

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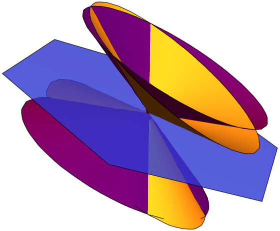
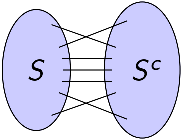
Idea: $q_1(\mathbf{x}) = \mathbf{x}^T Q_1 \mathbf{x}$ and $q_2(\mathbf{x}) = \mathbf{x}^T Q_2 \mathbf{x}$ are ≥ 0 on $\mathbb{R}^n_+$ and ≤ 0 on some hyperplane H, then so is $\lambda q_1 + \mu q_2$ for $\lambda, \mu \in \mathbb{R}_{\geq 0}$.



The second eigenvalue and expansion

Expansion of a graph $G = ([n], E)$:

$$h(G) = \min_{S \subseteq [n]} \frac{|E(S, S^c)|}{\min\{|S|, |S^c|\}}$$



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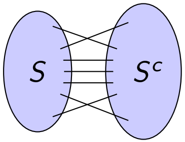

Cheeger's inequality: For any d-regular graph $G = ([n], E)$,

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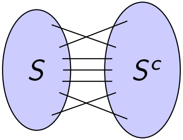

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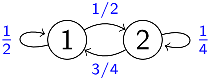
Conjecture (Mihail and Vazirani)
The edge graph of any 0-1 polytope has expansion ≥ 1.

The second eigenvalue and Markov Chains

A Markov chain on $[n] = \{1, \ldots, n\}$ is determined by a transition matrix $P \in \mathbb{R}_{\geq 0}^{n \times n}$ where P_{ij} represents $\mathrm{Prob}(i \to j)$.

Example: $P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$



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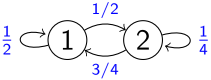

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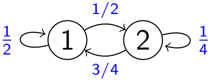

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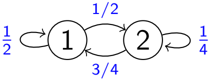

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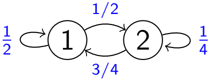

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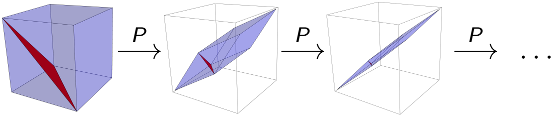

High dimensional expanders and random walks

Building on work of Dinur, Garland, Kaufman, Lubotzky, Mass, Oppenheim . . .

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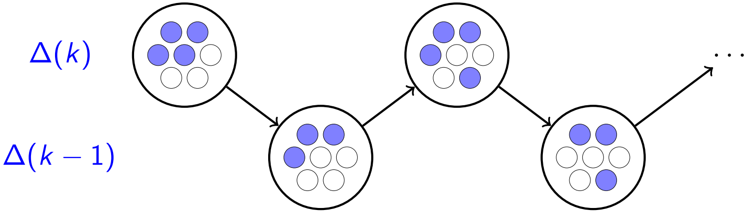
Random walks on $\Delta(k)$ and $\Delta(k-1)$

- $\sigma \in \Delta(k) \to \sigma \setminus \{i\}$, uniformly over $i \in \sigma$
- $\tau \in \Delta(k-1) \to \tau \cup \{j\}$ with prob. prop. to some weight



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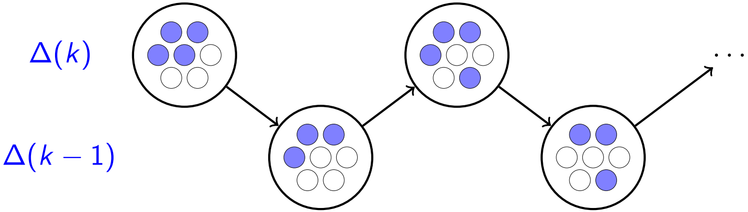
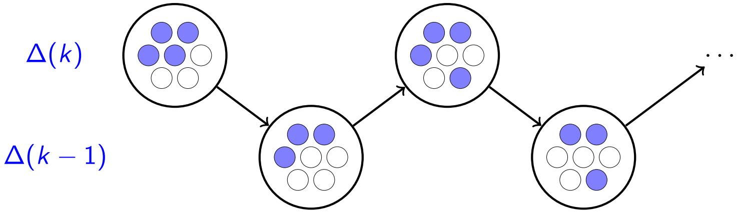


Down-up walk on $\Delta(k)$: transition matrix $P_k^{\vee}$

Up-down walk on $\Delta(k-1)$: transition matrix $P_{k-1}^{\wedge}$

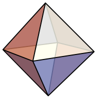
High dimensional expanders – local to global

Building on work of Dinur, Garland, Kaufman, Lubotzky, Mass, Oppenheim ...



Kaufman, Oppenheim (2018) bound the eigenvalues for the random walk on $\Delta(d)$ in terms of the eigenvalues of walks on the *links* of Δ.

$\rightarrow$ "λ-local spectral expander"



Anari, Liu, Oveis Gharan, V., (2019): The independence complex of any matroid is a 0-local spectral expander and the edge-graph of the matroid basis polytope has edge expansion ≥ 1.

Mihail, Vazirani (1989) conjecture this to hold for all 0-1 polytopes.

A small example

Random process on $[4] = \{1, 2, 3, 4\}$

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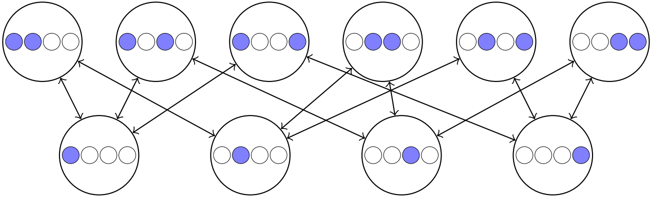

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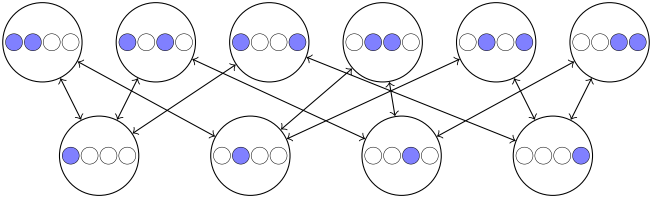

$$P_1^{\wedge} = \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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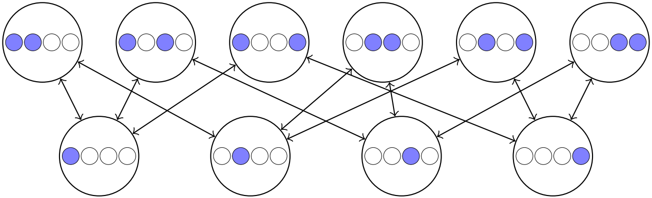

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$$= \frac{1}{2}\mathrm{Id}_4 + \frac{1}{6}\nabla^2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$$

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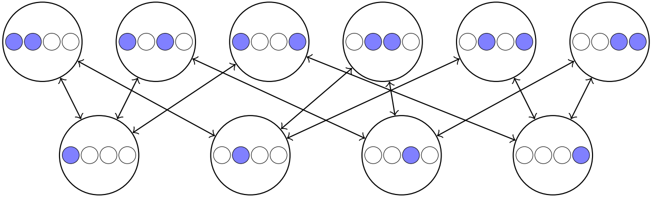
- $\{i,j\} \to \{i,j\}\backslash\{k\}$ uniformly over $k \in \{i,j\}$ (prob $= 1/2$)



$$P_1^{\wedge} = \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/2 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\mathrm{Id}_4 + \frac{1}{6}\nabla^2(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4)$$

$\Rightarrow \quad 0 \leq \lambda_n$ and $\lambda_2 \leq 1/2 \quad \Rightarrow \quad \lambda^*(P_1^{\wedge}) \leq 1/2$

A small example (continued)

Random process on $\binom{[4]}{2} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

- $\{i,j\} \to \{i,j\} \setminus \{k\}$ uniformly over $k \in \{i,j\}$ (prob $= 1/2$)
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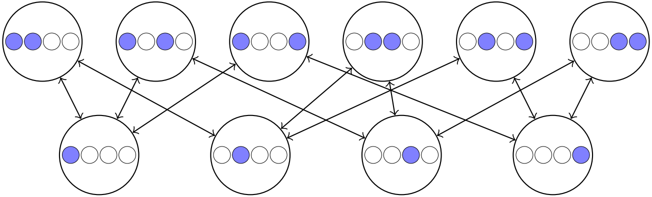

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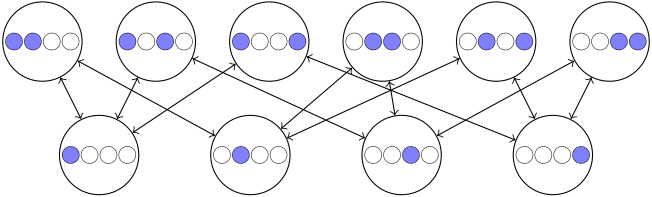

$$P_2^{\vee} = \begin{pmatrix} 1/3 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\ 1/6 & 1/3 & 1/6 & 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 1/3 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 1/3 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 & 1/6 & 1/3 & 1/6 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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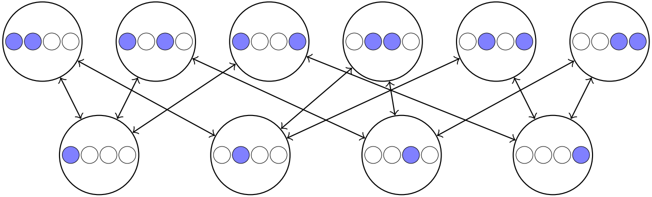

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$P_2^{\vee}$ has the same nonzero eigenvalues as $P_1^{\wedge}$ $\Rightarrow$ $\lambda^*(P_2^{\vee}) \leq 1/2$

Local spectral expanders

Building on work of Dinur, Garland, Kaufman, Lubotsky, Mass, Oppenheim ...

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(Idea)

- $P_k^\vee$ and $P_{k-1}^\wedge$ have the same nonzero eigenvalues (almost)
- Using connectivity and eig. val. on links, one can bound the eigenvalues of $P_k^\wedge$ as a function of the eigenvalues of $P_k^\vee$.

Translation to polynomials

$$(\Delta, w) \qquad \leftrightarrow \qquad f = \sum_{\sigma \in \Delta(d)} w(\sigma)\mathbf{x}^{\sigma}$$

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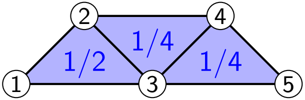

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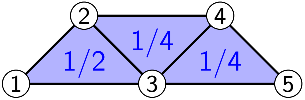

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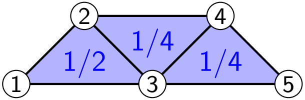

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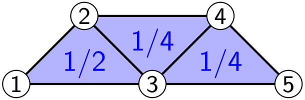

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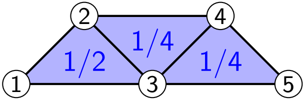

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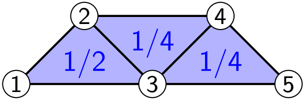	$\leftrightarrow$	$\frac{1}{2}x_1x_2x_3 + \frac{1}{4}x_2x_3x_4 + \frac{1}{4}x_3x_4x_5$
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Theorem (Anari, Liu, Oveis Gharan, V., 2019)
(Δ, w) is a 0-local spectral expander $\Leftrightarrow$ f is strongly log-concave.

Other consequences

Anari, Liu, Oveis Gharan, V., (2019): For any matroid with bases $\mathcal{B}$ and rank r, the down-up walk on $\mathcal{B}$ has mixing time $O(r^2 \log(n))$.

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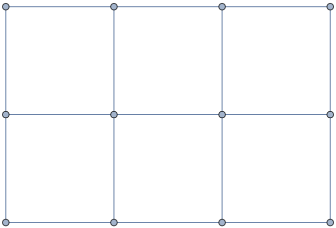
Improves on $n^{1+o(1)}$ Schild (2018), and many other previous works Aldous (1990), Broder (1989), Durfee, Kyng, Peebles, Rao, Sachdeva (2017)

Example: spanning trees of a grid

How can we connect the nodes in a 3×4 grid with the fewest possible connections?



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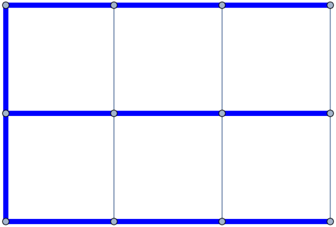

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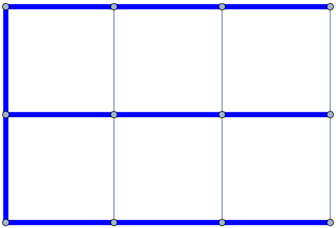
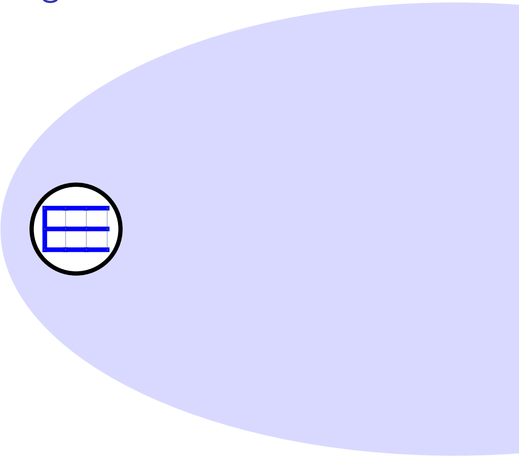
Idea: walk around space of possibilities by adding and removing redundant connections





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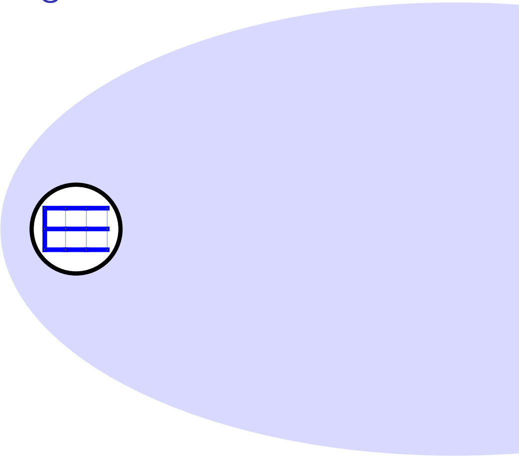

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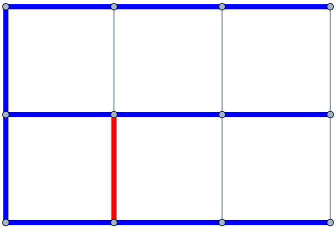

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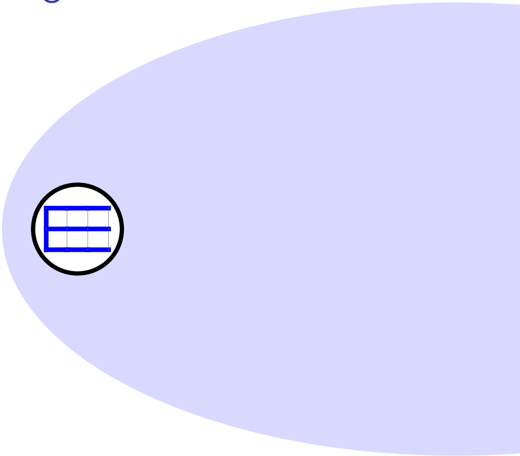

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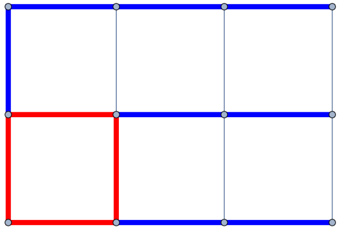

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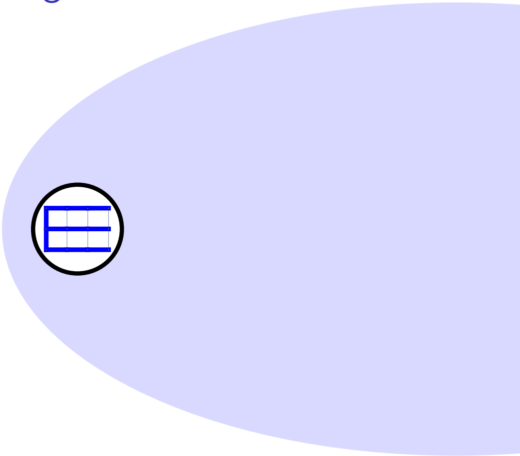

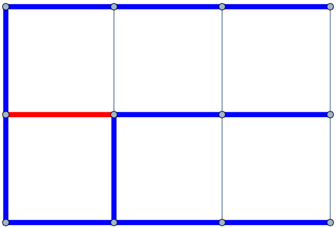

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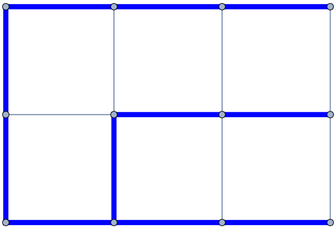

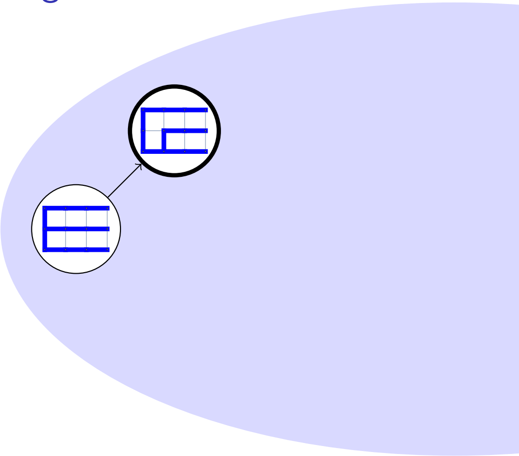

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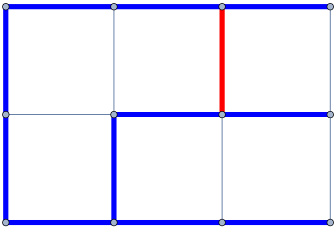

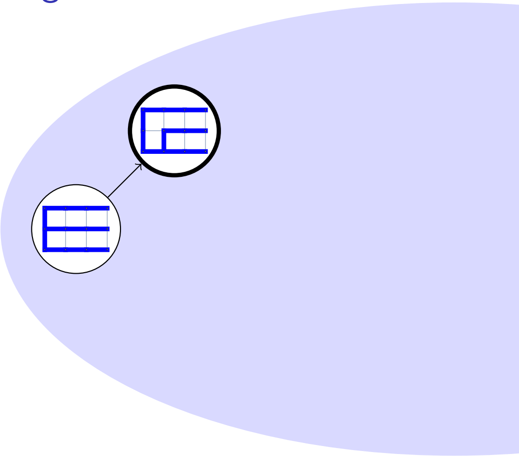

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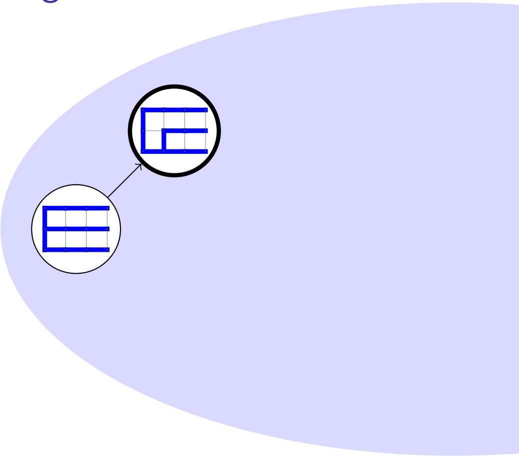

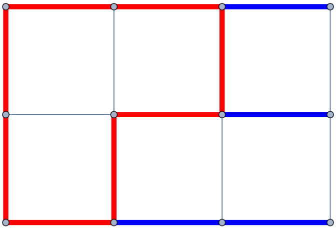

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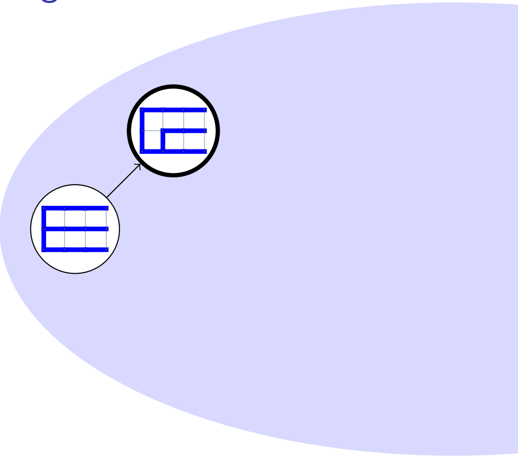

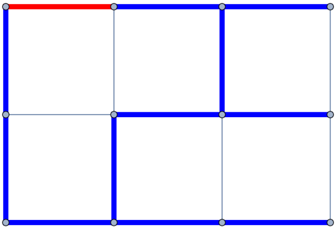

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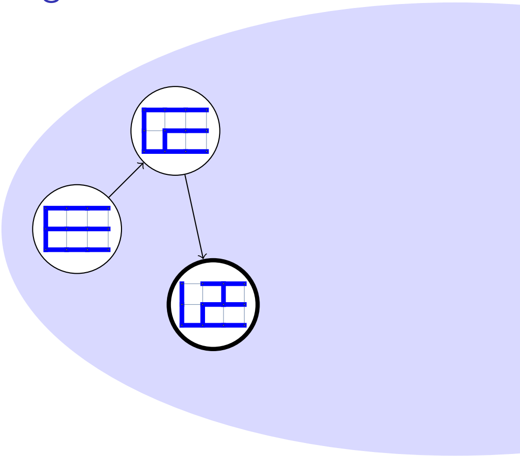

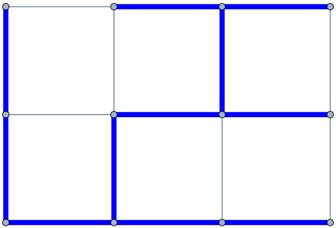

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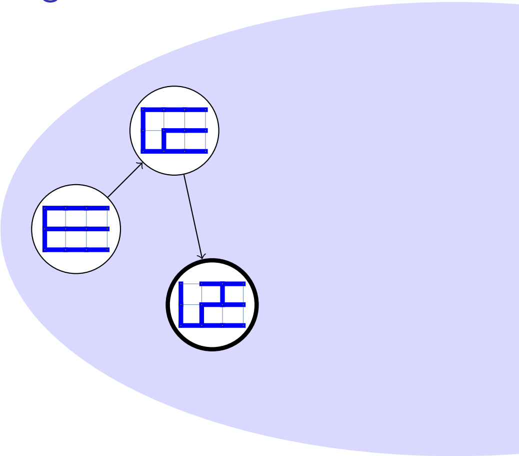

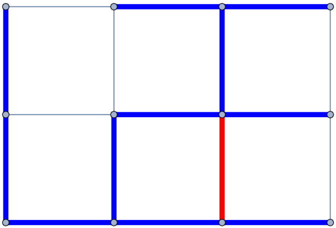

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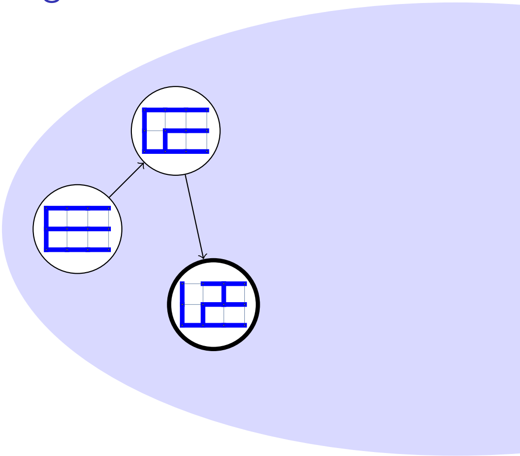
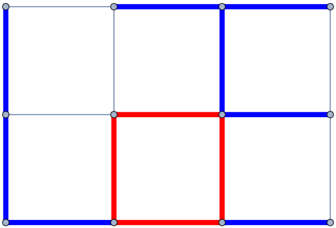
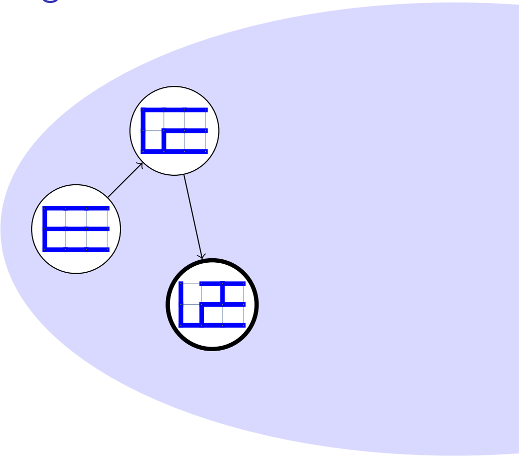




Example: spanning trees of a grid



How can we connect the nodes in a 3×4 grid with the fewest possible connections?

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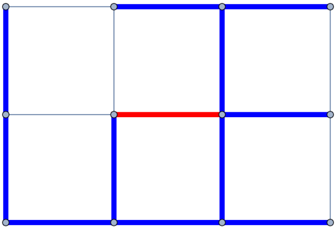

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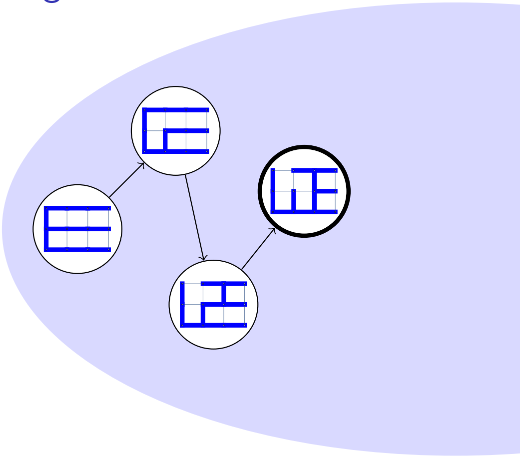

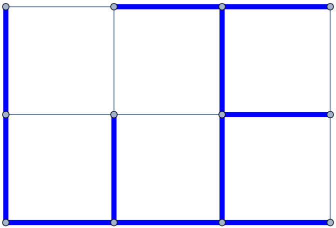

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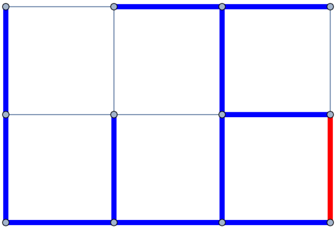

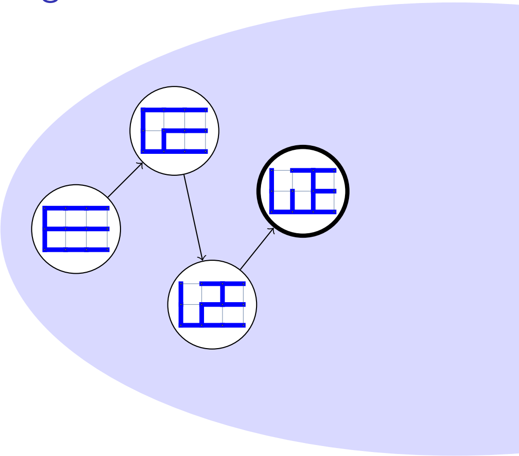

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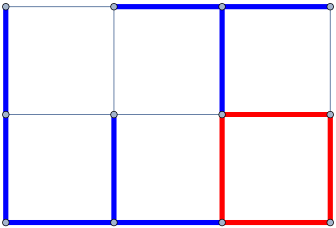

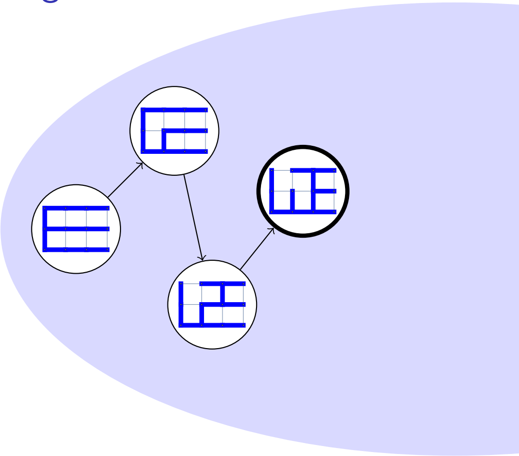

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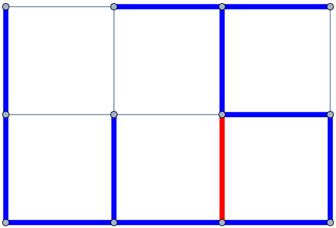

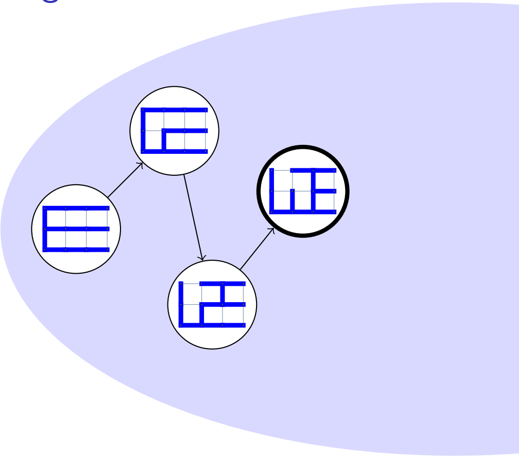

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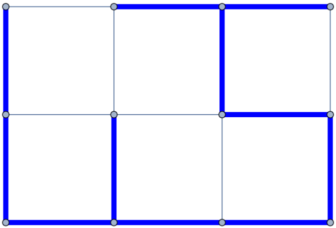

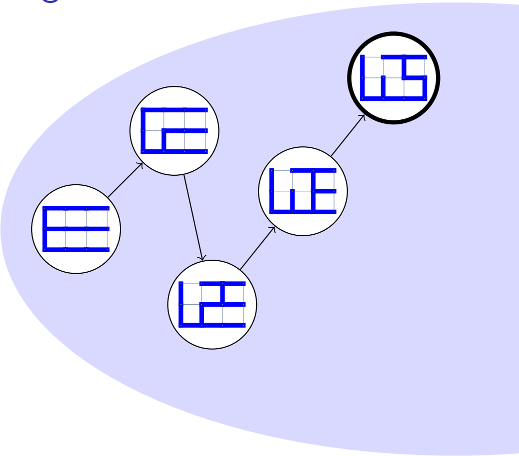

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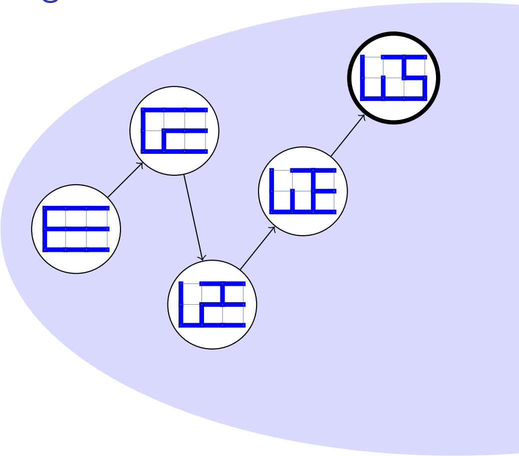

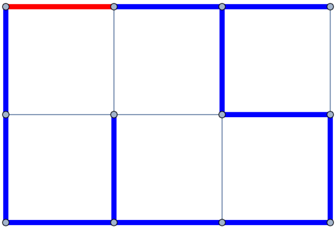

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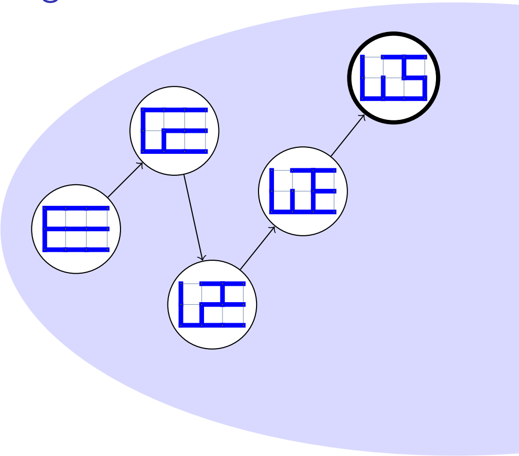

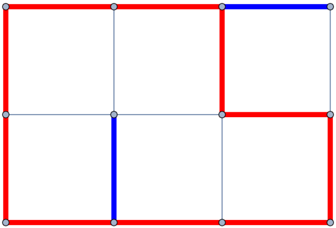

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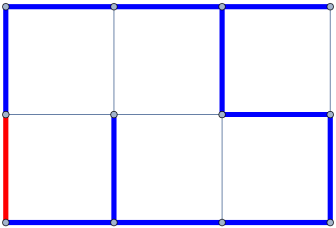

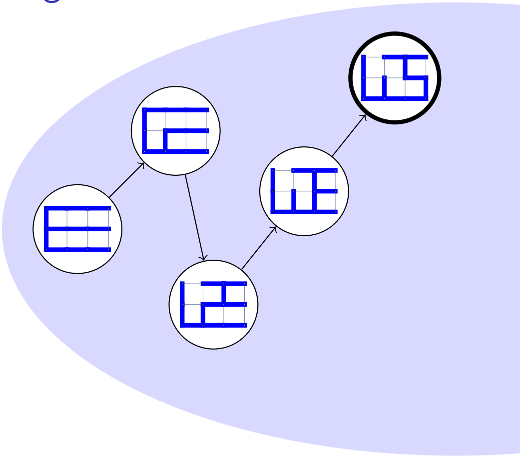

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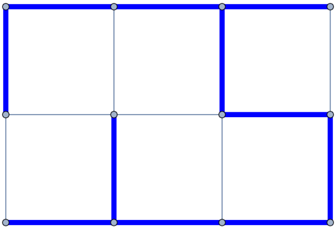

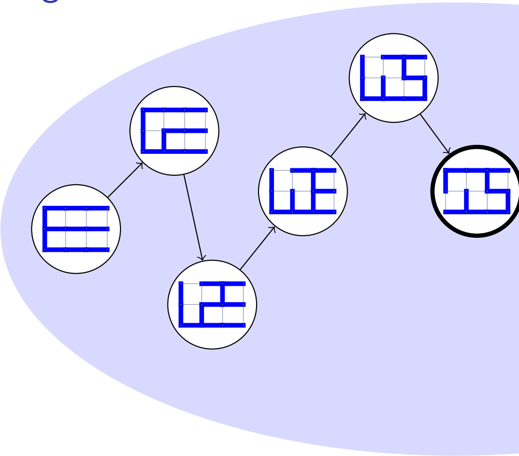

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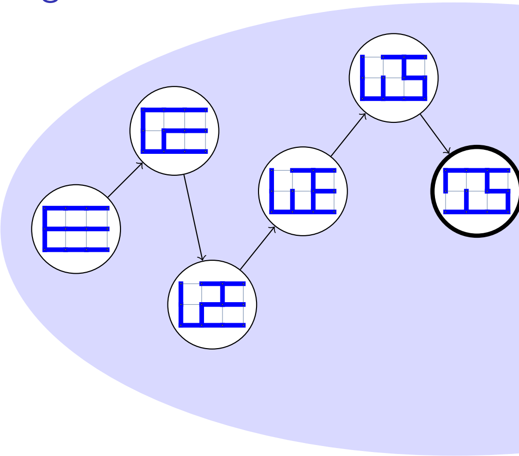

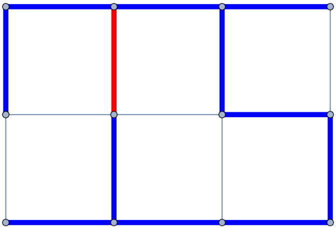

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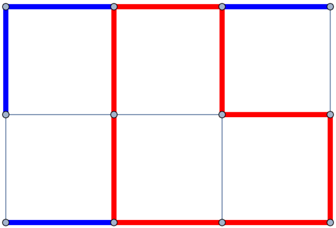

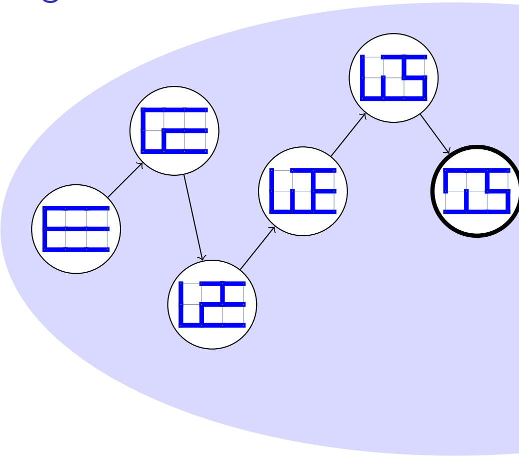

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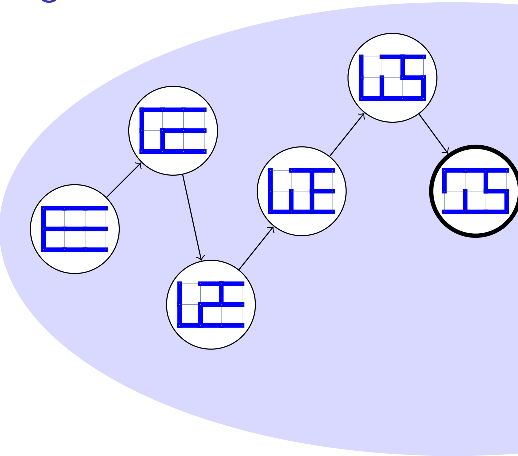

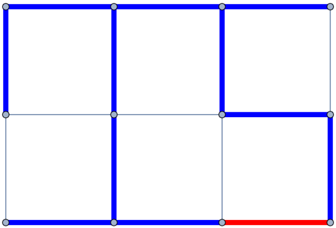

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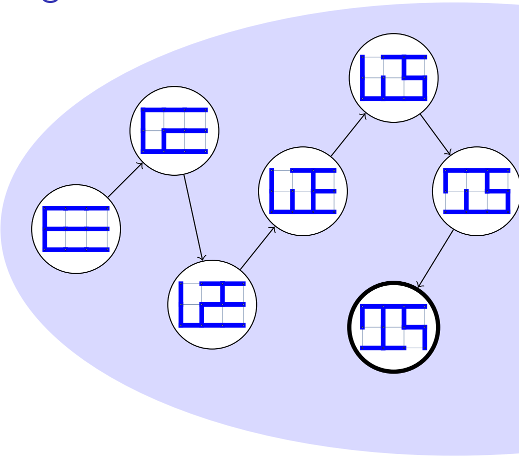

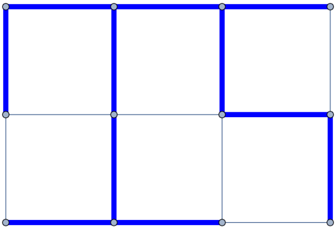

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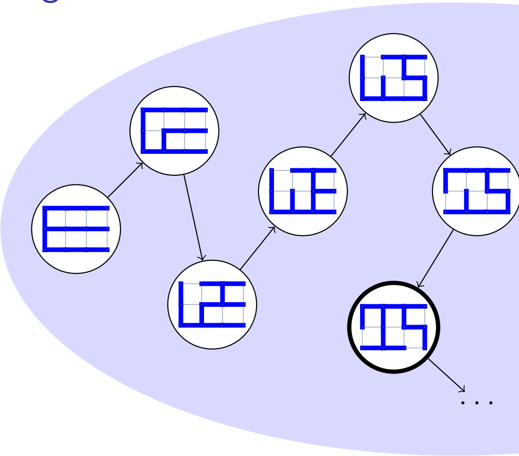

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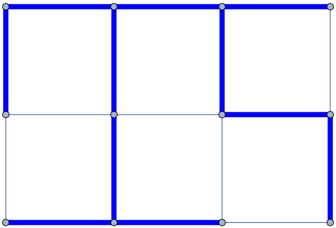

After 30 steps, every configuration is (about) equally likely (1 in 2415) no matter how we start.

Further Directions

Fractional log-concavity

Gen. poly. of λ-local spectral expanders are *fractionally log-concave*.

$(\lambda = 0)$ $\{$0-local spectral expanders$\}$ $=$ $\{$indep. complexes of matroids$\}$

$(\lambda > 0)$ $\{\lambda$-local spectral expanders$\}$ $=$???

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More general: spectral independence

Anari, Liu, Oveis Gharan (2020) use eigenvalues of correlation matrices to bound mixing time Glauber dynamics on distribution.

Abdolazimi, Liu, Oveis Gharan (2021): approximately sampling random proper edge colorings via rapid mixing

Zongchen Chen, Kuikui Liu, Eric Vigoda (2021): improve Barvinok's polynomial interpolation method, approximately sample for weighted edge cover problem and ferromagnetic Ising model in bounded degree

Conclusions

- strong log-concavity is a useful, testable condition
- connects discrete and functional log-concavty
- many interesting polynomials have this property, including matroid polynomials
- correspond to (0-local spectral) high dimensional expanders and implies rapid mixing of related Markov chains