Asymptotic behavior of solutions to the extension problem for the fractional Laplacian on hyperbolic spaces

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Contents

Fractional Laplacian on $\mathbb{R}^n$

- Fractional powers of the Laplace operator $(-\Delta)^\sigma$, $0 < \sigma < 1$: defined via Fourier transform

$$\mathcal{F}((-\Delta)^\sigma f)(\xi) = |\xi|^{2\sigma}(\mathcal{F}f)(\xi).$$

- Pointwise formula

$$(-\Delta)^\sigma f(x) = c_{n,\sigma} \text{ P.V. } \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x-z|^{n+2\sigma}} \, \mathrm{d}z.$$

- Nonlocal operator!

Fractional Laplacian on $\mathbb{R}^n$: extension problem

- Caffarelli-Silvestre:

Extension problem

$$\Delta v + \frac{(1-2\sigma)}{t}\partial_t v + \partial_{tt}^2 v = 0, \quad v(\cdot, 0) = f, \quad t > 0.$$

Then

$$(-\Delta)^\sigma f(x) = -2^{2\sigma-1}\frac{\Gamma(\sigma)}{\Gamma(1-\sigma)}\lim_{t\to 0^+} t^{1-2\sigma}\partial_t v(t,x).$$

(Dirichlet-to-Neumann)

- Fundamental kernel exists, can be computed explicitly:

$$Q_t^\sigma(x) = C_{n,\sigma}\frac{t^{2\sigma}}{(t^2+|x|^2)^{\sigma+\frac{n}{2}}},\ x \in \mathbb{R}^n,\ t > 0.$$

Fractional Laplacian on Riemannian mfds: extension problem

- Stinga-Torrea: general approach:

$$(-\Delta)^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-u(-\Delta)} - \mathrm{Id}) \frac{\mathrm{d}u}{u^{1+\sigma}}, \quad 0 < \sigma < 1.$$

  Connects problem with the heat semigroup!

- Banica-González-Sáez: On "*good*" noncompact complete manifolds $\mathcal{M}$, i.e. where given $x \in \mathcal{M}$, $\exists C_x > 0$, $\varepsilon > 0$ s.t. heat kernel h_t satisfies

$$\|h_t(x,\cdot)\|_{L^2(\mathcal{M})} + \|\partial_t h_t(x,\cdot)\|_{L^2(\mathcal{M})} \leq C_x(1+t^\varepsilon)t^{-\varepsilon},$$

  there exists a fundamental solution Q_t^σ to the extension problem.

Fractional Laplacian on Riemannian mfds: extension problem

- Solution to extension problem:

$$v(x,t) = \int_{\mathcal{M}} Q_t^\sigma(x,y) f(y) \, \mathrm{d}y,$$

where Q_t^σ is *fractional Poisson kernel*:

$$Q_t^\sigma(x,y) = \frac{t^{2\sigma}}{2^{2\sigma}\Gamma(\sigma)} \int_0^{+\infty} h_u(x,y)\, e^{-\frac{t^2}{4u}} \frac{\mathrm{d}u}{u^{1+\sigma}}.$$

- Examples of "good" mfds: Cartan-Hadamard mfds, Ric≥ 0.
- Such "good" mfds are stochastically complete, i.e. $\int_{\mathcal{M}} h_t(x,y)\mathrm{d}\mu(y) = 1$. Thus also $\int_{\mathcal{M}} Q_t^\sigma(x,y)\mathrm{d}\mu(y) = 1$.

Fractional Laplacian on Riemannian mfds: extension problem

- On $\mathbb{R}^n$ for $\sigma = 1/2 \rightsquigarrow$ Kernel of Poisson operator $e^{-t\sqrt{-\Delta}}$:

$$Q_t^{1/2}(x) = \frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}.$$

Asymptotics for Poisson operator on $\mathbb{R}^n$ (Vázquez)

Let $f \in L^1(\mathbb{R}^n)$ and $M := \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$. Then

$$\| e^{-t\sqrt{-\Delta}} f - M\, Q_t^{1/2} \|_{L^1(\mathbb{R}^n)} \to 0 \quad \text{as} \quad t \to +\infty.$$

- Question: Convergence true for $\sigma \in (0,1)$, different geometry?

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Interlude: Heat equation on $\mathbb{R}^n$

Heat equation

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) & \forall\, t>0,\ \forall\, x\in\mathbb{R}^n, \\ u(0,x) = f(x) & \forall\, x\in\mathbb{R}^n. \end{cases}$$

Heat kernel

$$h_t(x) = (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{\|x\|^2}{4t}}.$$

Asymptotics

Let $f\in L^1(\mathbb{R}^n)$ and $M := \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$. Then

$$\|u(t,\cdot) - M\, h_t\|_{L^1} \to 0 \quad \text{as} \quad t\to+\infty.$$

Interlude: Heat equation on Riemannian mfds of Ric ≥ 0

- $\mathcal{M}$: complete, connected, noncompact Riemannian manifold of nonnegative Ricci curvature
- Volume is doubling, i.e., for all $x \in \mathcal{M}$ and $r > 0$, we have

$$V(x, 2r) \leq C\, V(x, r).$$

Two-sided estimates of the heat kernel [Li-Yau (1986)]

$$\frac{c_1}{V(x, \sqrt{t})}\, e^{-C_1 \frac{d^2(x,y)}{t}} \leq h_t(x, y) \leq \frac{C_2}{V(x, \sqrt{t})}\, e^{-c_2 \frac{d^2(x,y)}{t}}$$

[Grigor'yan-P.-Zhang (2023)]

For $\mathcal{M}$ as above, fix a base point $x_0 \in \mathcal{M}$ and assume initial data $f \in L^1(\mathcal{M})$. Then as $t \to +\infty$,

$$\|u(t, \cdot) - M\, h_t(\cdot, x_0)\|_{L^1(\mathcal{M})} \to 0.$$

Hyperbolic space

- Complete Riemannian manifold, simply connected, sectional curvature -1
- **Model:** hyperboloid
$$\{x \in \mathbb{R}^{n+1} \,|\, x_1^2 + \ldots + x_n^2 - x_0^2 = -1, x_0 \geq 1\}$$
hyperbolic metric $ds^2 = dx_1^2 + \ldots + dx_n^2 - dx_0^2\big|_{T_x\mathbb{H}^n}$
- Polar coordinates: $x = (\cosh r, \sinh r\, \omega)$, $r > 0$, $\omega \in \mathbb{S}^{n-1}$ $\rightsquigarrow$ $r = d(x, o)$ distance to origin $o = (1, 0, ..., 0)$
- Distance between two arbitrary points:
$$d(x, x') = \cosh^{-1}(\cosh r \cosh r' - \sinh r \sinh r'\, \omega \cdot \omega')$$
- $d\mathrm{vol} = c_n \sinh^{n-1} r\, dr\, d\omega$
- $\mathbb{H}^n = G/K$, $G = SO^0(n, 1)$, $K = SO(n)$ $\rightsquigarrow$ symmetric space of noncompact type and rank one

Heat kernel on $\mathbb{H}^n$

Upper/lower bounds [Davies-Mandouvalos (1988), Anker-Ji (1999), Anker-Ostellari (2003)]

$$h_t(x,y) \asymp t^{-\frac{n}{2}}(1+r)(1+t+r)^{\frac{n-3}{2}}\, e^{-(\frac{n-1}{2})^2 t - \frac{n-1}{2} r - \frac{r^2}{4t}}$$

for every $t>0$ and for every $x,y \in \mathbb{H}^n$, where $r = d(x,y)$.

Asymptotics [Vázquez (2019)], [Anker-P.-Zhang (2023)]

Let $f \in L^1(\mathbb{H}^n)$, $M := \int_{\mathbb{H}^n} f$. Assume f is radial, i.e., $f(x)$ depends only on $r = d(x,o)$. Then

$$\|e^{-t(-\Delta)}f - M\,h_t(\cdot,o)\|_{L^1(\mathbb{H}^n)} \to 0 \quad \text{as} \quad t \to +\infty.$$

Moreover, this result may fail if f is not radial.

Heat kernel on $\mathbb{H}^n$

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Fourier analysis on $\mathbb{H}^n = G/K$ [Harish–Chandra, Helgason]

- For $x = (r, \omega) \in \mathbb{H}^n$, $\lambda \in \mathbb{R}$, $\theta \in \mathbb{S}^{n-1}$, define

$$A(x, \theta) := \log(\cosh r - \sinh r\, \omega \cdot \theta)$$

$$e_{\lambda,\theta}(x) := \exp\left\{\left(i\lambda - \frac{n-1}{2}\right) A(x, \theta)\right\}$$

$$\varphi_\lambda(x) := \int_{\mathbb{S}^{n-1}} e_{\lambda,\theta}(x)\, \mathrm{d}\theta = \text{elementary spherical function of index } \lambda$$

- Properties: φ_λ radial, $\varphi_\lambda = \varphi_{-\lambda}$, $\lambda \in \mathbb{C}$
- Helgason-Fourier transform

$$\mathcal{H}f(\lambda, \theta) = \int_{\mathbb{H}^n} f(x)\, \mathrm{e}_{\lambda,\theta}(x)\, \mathrm{d}\mu(x)$$

- Spherical Fourier transform (of radial functions)

$$\mathcal{H}f(\lambda, \theta) = \mathcal{H}f(\lambda) = \int_{\mathbb{H}^n} f(x)\, \varphi_\lambda(x)\, \mathrm{d}\mu(x)$$

Return to extension problem on $\mathbb{H}^n$

- For any $\sigma \in (0,1)$, solution to

$$\Delta v + \frac{(1-2\sigma)}{t}\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial t^2} = 0, \quad v(0,x) = f(x), \quad t>0, \; x \in \mathbb{H}^n,$$

  is given by

$$v(t,x) = \int_{\mathbb{H}^n} Q_t^\sigma(x,y)\, f(y)\, \mathrm{d}\mu(y).$$

- Q_t^σ radial, by subordination to (radial) heat kernel.
- Upper and lower bounds [Bhowmik-Pusti (2022)]:

$$Q_t^\sigma(r) \asymp \begin{cases} t^{2\sigma}(t^2+r^2)^{-\frac{n}{2}-\sigma}, & t^2+r^2<1 \\ t^{2\sigma}(t^2+r^2)^{-1-\frac{\sigma}{2}}(1+r)e^{-\frac{n-1}{2}r}e^{-\frac{n-1}{2}\sqrt{t^2+r^2}}, & t^2+r^2 \geq 1 \end{cases}$$

Large-time asymptotics in $L^1(\mathbb{H}^n)$ for extension problem

[P. (2024)]

Let $f \in L^1(\mathbb{H}^n)$ be radial, v solution to extension problem with initial data f. Set $M := \int_{\mathbb{H}^n} f$. Then

$$\|v(t,\cdot) - M\, Q_t^{\sigma}(\cdot, o)\|_{L^1(\mathbb{H}^n)} \longrightarrow 0 \qquad \text{as} \quad t \to +\infty.$$

Convergence fails in general without radiality assumption (counterexample: any solution $Q_t^{\sigma}(\cdot, y)$, $y \neq o$).

- Result extends to all noncompact symmetric spaces of arbitrary rank.

Large-time asymptotics in $L^1(\mathbb{H}^n)$ for extension problem

Fractional Poisson kernel concentration: critical region [P. (2024)]

For any $\sigma \in (0,1)$, the fractional Poisson kernel Q_t^σ concentrates asymptotically in the annulus

$$\Omega_t = \{x \in \mathbb{H}^n \,|\, t^{2-\varepsilon} \le d(x,o) \le t^{2+\varepsilon}\},$$

$0 < \varepsilon < 2$, in the sense that

$$\int_{\Omega_t} Q_t^\sigma(x)\, \mathrm{d}\mu(x) \to 1, \quad \text{i.e.} \quad \int_{\mathbb{H}^n \setminus \Omega_t} Q_t^\sigma(x)\, \mathrm{d}\mu(x) \to 0.$$

- Comparison: critical region for Q_t^σ on $\mathbb{R}^n$: $\Omega_t = \{x \in \mathbb{R}^n \,|\, t^{1-\varepsilon} \le d(x,o) \le t^{1+\varepsilon}\}$.
- For h_t on $\mathbb{H}^n$: $\Omega_t = \{x \in \mathbb{H}^n \,|\, (n-1)t - t^{1/2+\varepsilon} \le d(x,o) \le (n-1)t + t^{1/2+\varepsilon}\}$.

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Sketch of the proof

- By density, we may assume that $f \in C_c(\mathbb{H}^n)$.
  Assume $\operatorname{supp} f \subseteq \{x \in \mathbb{H}^n : \ d(x,o) < \xi\}$.
- Outside the critical region Ω_t : On the one hand,
  $$\int_{\mathbb{H}^n \smallsetminus \Omega_t} Q_t^\sigma(x)\,\mathrm{d}\mu(x) \to 0.$$
  On the other hand, control L^1 norm of
  $$v(t,x) = \int_{\mathbb{H}^n} Q_t^\sigma(x,y)\,f(y)\,\mathrm{d}\mu(y)$$
  by reduction to fractional Poisson kernel asymptotics to deduce that
  $$\|v(t,\cdot)\|_{L^1(\mathbb{H}^n \smallsetminus \Omega_t)} \to 0.$$
- Inside the critical region Ω_t : Use fractional Poisson kernel large-time asymptotics for the difference $v(t,x) - M\,Q_t^\sigma(x)$.

Estimate outside the critical region

- We have $v(t,x) = \int_{B(o,\xi)} Q_t^\sigma(x,y) f(y)\,\mathrm{d}\mu(y)$, therefore

$$\|v(t,\,\cdot\,)\|_{L^1(\mathbb{H}^n \smallsetminus \Omega_t)} \leq \int_{B(o,\xi)} |f(y)| \int_{\mathbb{H}^n \smallsetminus \Omega_t} Q_t^\sigma(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

- If $x \in \mathbb{H}^n \smallsetminus \Omega_t$ then $x \in \mathbb{H}^n \smallsetminus \widetilde{\Omega_{t,y}}$, where

$$\widetilde{\Omega_{t,y}} = \left\{ x \in \mathbb{H}^n \,|\, 2t^{2-\varepsilon} \leq d(x,y) \leq \frac{1}{2} t^{2+\varepsilon} \right\}.$$

- Therefore,

$$\int_{\mathbb{H}^n \smallsetminus \Omega_t} Q_t^\sigma(x,y)\,\mathrm{d}\mu(x) \leq \int_{\mathbb{H}^n \smallsetminus \widetilde{\Omega_{t,y}}} Q_t^\sigma(x,y)\,\mathrm{d}\mu(x).$$

Estimate outside the critical region

- By fractional Poisson kernel estimates

$$\|v(t,\cdot)\|_{L^1(\mathbb{H}^n\smallsetminus\Omega_t)} \lesssim \|f\|_{L^1(\mathbb{H}^n)}\, t^{-\sigma\varepsilon} \quad \forall t>1.$$

- Altogether,

$$\|v(t,\cdot)-M\,Q_t^\sigma\|_{L^1(\mathbb{H}^n\smallsetminus\Omega_t)} \leq \|u(t,\cdot)\|_{L^1(\mathbb{H}^n\smallsetminus\Omega_t)}+M\|Q_t^\sigma\|_{L^1(\mathbb{H}^n\smallsetminus\Omega_t)} \to 0.$$

Critical region: fractional Poisson kernel asymptotics

- Write

$$v(t,x) - M\,Q_t^\sigma(x) = \int_G (Q_t^\sigma(x,y) - Q_t^\sigma(x))\,f(y)\,\mathrm{d}\mu(y)$$
$$= Q_t^\sigma(x)\int_{B(o,\xi)} \left(\frac{Q_t^\sigma(x,y)}{Q_t^\sigma(x)} - 1\right)\,f(y)\,\mathrm{d}\mu(y).$$

- **Aim :** Find asymptotics for the quotient

$$\frac{Q_t^\sigma(d(x,y))}{Q_t^\sigma(d(x,o))}$$

for $x \in \Omega_t$ and $y \neq o$, $d(y,o) < \xi$.

Critical region: fractional Poisson kernel asymptotics

[P. (2024)]

$$Q_t^\sigma(r) \sim C(\sigma)\, t^{2\sigma}\, \gamma\left(\frac{n-1}{2}\frac{r}{\sqrt{t^2+r^2}}\right) r\,(t^2+r^2)^{-1-\frac{\sigma}{2}} \times$$
$$\times\, e^{-\frac{n-1}{2}r-\frac{n-1}{2}\sqrt{t^2+r^2}}, \qquad \text{as } t+r \to +\infty.$$

[1]

- Quotient for large time, when $t^{2-\varepsilon} \leq d(x,o) \leq t^{2+\varepsilon}$, y bdd:

$$\frac{Q_t^\sigma(d(x,y))}{Q_t^\sigma(d(x,o))} \sim e^{\frac{n-1}{2}(d(x,o)-d(x,y))\left(1+\frac{d(x,o)+d(x,y)}{\sqrt{t^2+d^2(x,o)}+\sqrt{t^2+d^2(x,y)}}\right)}$$

- As $t \to +\infty$: Blue term $\to 1$; $d(x,o)-d(x,y) \to ?$

[1] $\gamma(s) = \frac{\Gamma(s+1/2)\Gamma(s/2+(n-1)/4)}{\Gamma(s+1)\Gamma(s/2+1/4)}$

Critical region: fractional Poisson kernel asymptotics

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1

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Critical region: heat kernel asymptotics

[P. (2024)]

For $(r,\omega) = x \in \Omega_t$ and $y = (s,\theta)$ bounded,

$$d(x,o) - d(x,y) = \log(\cosh s - \sinh s\,\omega\cdot\theta) + \mathrm{O}(t^{-2+\varepsilon})$$
$$= A(y,\omega) + \mathrm{O}(t^{-2+\varepsilon}),$$

- In the case of $\mathbb{H}^n$, one can use known formula for $d(x,y)$
- In general LHS defines a Busemann function
- For higher rank symmetric spaces, compute using Iwasawa decomposition

Critical region: heat kernel asymptotics

[P. (2024)]

For $x \in \Omega_t$, y bounded and $t \to +\infty$,

$$\frac{Q_t^\sigma(x,y)}{Q_t^\sigma(x,o)} = e^{2\rho A(y,\omega)} + \mathrm{O}\big(t^{-2+\varepsilon}\big), \quad \rho = \frac{n-1}{2}.$$

- Recall Helgason-Fourier transform for $f \in C_c(\mathbb{H}^n)$:

$$\mathcal{H}f(\lambda,\omega) = \int_{\mathbb{H}^n} f(y) \exp\{(-i\lambda+\rho)A(y,\omega)\}\,\mathrm{d}\mu(y)$$

- By previous asymptotics

$$v(t,x) - M\,Q_t^\sigma(x) = Q_t^\sigma(x) \int_{\mathbb{H}^n} \left(\frac{Q_t^\sigma(x,y)}{Q_t^\sigma(x,o)} - 1 \right) f(y)\,\mathrm{d}\mu(y)$$

$$= \underbrace{Q_t^\sigma(x)}_{\int_{\Omega_t} Q_t^\sigma \to 1} \left(\mathcal{H}f(i\rho,\omega) - \mathcal{H}f(-i\rho,\omega) + \underbrace{\mathrm{O}\big(t^{-2+\varepsilon}\,\|f\|_1\big)}_{\to 0} \right).$$

Radial vs. non radial data

Radial data: convergence [P. (2024)]

If f is radial, then

$$\cancel{\mathcal{H}f(i\rho,\omega)} = \mathcal{H}f(i\rho) = \mathcal{H}f(-i\rho) = \cancel{\mathcal{H}f(-i\rho,\omega)} = \int_{\mathbb{H}^n} f,$$

thus

$$\| v(t,\cdot) - M\, Q_t^{\sigma} \|_{L^1(\mathbb{H}^n)} \xrightarrow{t\to+\infty} 0.$$

Non radial data: counterexample [P. (2024)]

Take $y \neq o$ initial data $f = \delta_y \rightsquigarrow$ solution: displaced kernel

$$v(t,x) = Q_t^{\sigma}(x,y)$$

Then

$$\| Q_t^{\sigma}(\cdot,y) - Q_t^{\sigma}(\cdot,o) \|_{L^1(\Omega_t)} \xrightarrow{t\to+\infty} \int_{\mathbb{S}^{n-1}} \left| e^{2\rho A(y,\omega)} - 1 \right| \mathrm{d}\omega$$

which is > 0 if $y \neq o$.

THANK YOU
FOR YOUR ATTENTION!

Details on higher rank

Critical region on higher rank

Let $0 < \varepsilon < 1$. Consider in $\mathfrak{a}$ the annulus

$$t^{2-\varepsilon} \leq |H| \leq t^{2+\varepsilon}$$

and the solid cone $\Gamma(t)$ with angle

$$\gamma(t) = t^{-\frac{\varepsilon}{2}}$$

around the ρ-axis, and denote by Ω_t their intersection. Then, the critical region for the fractional Poisson kernel is $K(\exp \Omega_t)K$, in the sense that

$$\int_{G \setminus K(\exp \Omega_t)K} Q_t^{\sigma}(g)\, \mathrm{d}g \longrightarrow 0, \quad \text{as } t \to +\infty.$$

Details on higher rank

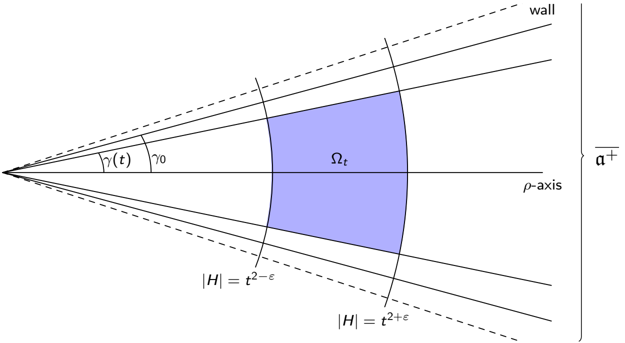

Figure: Flat part Ω_t of critical region

Details on higher rank

Lemma

Let $x \in K(\exp \Omega_t)K$ *and let* y *be bounded. Then*

$$\langle \rho, x^+ \rangle - \langle \rho, (y^{-1}x)^+ \rangle = |\rho||x^+| - |\rho||(y^{-1}x)^+| + O(t^{-\frac{\varepsilon}{2}}).$$

- Then one can use a calculation for the Busemann function relying on the Iwasawa decomposition