Strichartz estimates in the Heisenberg group

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Joint work with

Based on two joint works with

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and

- Steven Flynn (Univ. Padova)

→ Main references:

BBG-21 H.Bahouri, D.Barilari, I.Gallagher, *Strichartz estimates and Fourier restriction theorems in the Heisenberg group*, JFAA, 2021

BF-24 D.Barilari, S.Flynn *Strichartz estimates on H-type Carnot groups*, in preparation

The Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

From the explicit expression of the solution, using Fourier analysis:

$$u(t,\cdot) = \frac{e^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0 \,.$$

one obtains the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \quad (1)$$

The TT^* argument

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Once one has the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|u_0\|_{L^1(\mathbb{R}^n)} \tag{2}$$

together with the conservation of the L^2 norm ($\rightarrow \widehat{u}(t,\xi) = e^{it|\xi|^2}\widehat{u}_0(\xi)$)

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \tag{3}$$

one can obtain interpolating estimates in L^p spaces

Strichartz estimates

For the free Schrödinger one has the following estimate

Strichartz estimate

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)} , \tag{4}$$

where (p, q) satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \qquad q \geq 2,\ (n, q, p) \neq (2, 2, \infty)$$

$\rightarrow$ the necessity can be obtained by rescaling

Assume the following holds for every $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{5}$$

Give a solution $u = u(t,x)$ with $u(0,\cdot) = u_0$ then

- also, $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution
- with initial datum $u_{0,\lambda}(x) = u(0,\lambda x) = u_0(\lambda x)$

Let us compute the two sides for u_λ

- $\|u_\lambda\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} = \lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}$.
- $\|u_{0,\lambda}\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}$

One gets

$$\lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{6}$$

which forces the equality

The rescaling argument

Assume the following holds for every $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{7}$$

Give a solution $u = u(t,x)$ with $u(0,\cdot) = u_0$ then

- also, $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution
- with initial datum $u_{0,\lambda}(x) = u(0,\lambda x) = u_0(\lambda x)$

Let us compute the two sides for u_λ

- $\|u_\lambda\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} = \lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}$.
- $\|u_{0,\lambda}\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}$

One gets

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\lambda^{\frac{n}{2}-\frac{2}{q}-\frac{n}{p}}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{8}$$

which forces the equality

Strichartz estimates

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For the free Schrödinger one has the following estimate

Strichartz estimate

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C_{p,q} \|u_0\|_{H^\sigma(\mathbb{R}^n)}\,, \tag{9}$$

where (p, q) satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} \leq \frac{n}{2}, \qquad q \geq 2,\ (n, q, p) \neq (2, 2, \infty)$$

→ the necessity can be obtained by rescaling

→ here $\sigma = \frac{n}{2} - \frac{2}{q} - \frac{n}{p}$

Consequence

The dispersive inequality also yields the following Strichartz inequalities for the inhomogeneous Schrödinger equation $i\partial_t u - \Delta u = f$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\left(\|u_0\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{p}'}(\mathbb{R}^n))}\right), \quad (10)$$

- (p, q) and (p_1, q_1) satisfy the admissibility condition
- a' the dual exponent of any $a \in [1, \infty]$.
- crucial in the study of semilinear and quasilinear Schrödinger equations

An application : for small datum the cubic semilinear equation in $\mathbb{R}^2$

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u \\ u_{|t=0} = u_0\,, \end{cases}$$

has a solution in $L^\infty_t L^2_x \cap L^3_t L^6_x$

The Heisenberg group $\mathbb{H}$

$\mathbb{H} \sim \mathbb{R}^3$

$$X_1 := \partial_1 - \frac{x_2}{2}\partial_3 , \quad X_2 := \partial_2 + \frac{x_1}{2}\partial_3 , \quad X_3 := \partial_3 .$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \end{pmatrix}$$

- We have $[X_1, X_2] = X_3$
- the distribution $D = \mathrm{span}\{X_1, X_2\}$ is bracket generating
- it is also left-invariant
- homogeneous with respect to $\delta_\varepsilon(x_1, x_2, x_3) = (\varepsilon x_1, \varepsilon x_2, \varepsilon^2 x_3)$

$\mathbb{H} \sim \mathbb{R}^3$

$$X_1 := \partial_1 - \frac{x_2}{2}\partial_3 \ , \quad X_2 := \partial_2 + \frac{x_1}{2}\partial_3 \, , \quad X_3 := \partial_3 \, .$$

Group law:

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The Haar measure is equal to the Lebesgue measure.

Convolution product $f \star g(x) := \int_{\mathbb{H}} f(x \cdot y^{-1}) g(y)\, dy \, .$

Homogeneous dimension

$Q = \sum_j j \dim \mathfrak{g}_j = 4 \, , \quad \left|B_{\mathbb{H}}(x, r)\right| = r^Q \left|B_{\mathbb{H}}(0, 1)\right|$

Laplacian in Heisenberg

- the horizontal vector fields X and Y are defined by

$$X = \partial_x - \frac{y}{2}\partial_z, \qquad Y = \partial_y + \frac{x}{2}\partial_z\,.$$

- The horizontal gradient

$$\nabla_{\mathbb{H}} u = (Xu)X + (Yu)Y.$$

- Complex notations $Z = X + iY$ and $\bar{Z} = X - iY$

$$\Delta_{\mathbb{H}} u = (X^2 + Y^2)u = Z\overline{Z} - i\partial_z,$$

$\rightarrow$ non elliptic, hypelliptic

Remark (on Shrödinger equation in $\mathbb{H}$)

$$i\partial_t u - \Delta_{\mathbb{H}} u = 0 \quad \Leftrightarrow \quad i(\partial_t + \partial_z)u = Z\overline{Z}u$$

No dispersion in Heisenberg

The linear Schrödinger equations on $\mathbb{H}$ associated with the sublaplacian

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = u_0 \, , \end{cases}$$

Theorem (Bahouri-Gérard-Xu 2000)

There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H})$ such that the solution to the free Schrödinger equation satisfies

$$u(t, x_1, x_2, x_3) = u_0(x_1, x_2, x_3 + t) \, .$$

In particular for all $1 \leq p \leq \infty$

$$\|u(t, \cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

$\rightarrow$ no dispersion

Baouendi-Grushin operator

In $L^2 = L^2(\mathbb{R}^2, dxdy)$, consider the action of the Baouendi-Grushin operator

$$\Delta_G = \partial_x^2 + x^2\partial_y^2. \tag{11}$$

This operator is the Laplacian of the sub-Riemannian structure on $\mathbb{R}^2$ defined by

$$X = \partial_x, \quad Y = x\partial_y. \tag{12}$$

meaning that $\Delta_G = X^2 + Y^2$. Consider the associated Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \quad u(0, \cdot) = u_0. \tag{13}$$

No dispersion

The associated Schrödinger equation

$$i\partial_t u + \Delta_G u = 0, \quad u(t=0) = u_0. \tag{14}$$

is also nondispersive.

there exist initial data u_0 for which the solution u satisfies

$$\|u(t)\|_{L^p} = \|u_0\|_{L^p} \quad \forall t \in \mathbb{R}, \quad p \geq 1. \tag{15}$$

This phenomenon is due to a transport behaviour of Δ_{BG} in the vertical direction. Let us show this fact.

Baouendi-Grushin operator in Fourier

For any $u \in L^2$, write

$$u(x,y) = \int_{\mathbb{R}} e^{i\lambda y} \widehat{u}(x,\lambda) d\lambda,$$

where $\widehat{u}(x,\lambda)$ is the Fourier transform of u w.r.t. the y-variable.

$$\Delta_G u = \int_{\mathbb{R}} e^{i\lambda y} (\partial_x^2 - x^2\lambda^2) \widehat{u}(x,\lambda) d\lambda =: \int_{\mathbb{R}} e^{i\lambda y} \widehat{\Delta_G}(\lambda) \widehat{u}(x,\lambda) d\lambda,$$

where we defined the Hermite operator

$$\widehat{\Delta_G}(\lambda) = \partial_x^2 - x^2\lambda^2$$

for which we know eigenvalues and eigenfunctions.

Let $h_n(x)$ be the n^{th} Hermite function, which satisfies the ODE

$$\frac{d^2}{dx^2}h_n(x) - x^2 h_n(x) = -(2n+1)h_n(x),$$

then $h_n^\lambda(x) := h_n(\sqrt{|\lambda|}x)$ satisfies

$$\frac{d^2}{dx^2}h_n^\lambda(x) - x^2\lambda^2 h_n^\lambda(x) = -(2n+1)|\lambda| h_n^\lambda(x).$$

We can then write for any $\lambda \neq 0$

$$\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} \widehat{u}_n(\lambda) h_n^\lambda(x), \tag{16}$$

and obtain

$$\widehat{\Delta_G}(\lambda)\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} -(2n+1)|\lambda|\widehat{u}_n(\lambda)h_n^\lambda(x).$$

Let $h_n(x)$ be the n^{th} Hermite function, which satisfies the ODE

$$\frac{d^2}{dx^2}h_n(x) - x^2 h_n(x) = -(2n+1)h_n(x),$$

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We can then write for any $\lambda \neq 0$

$$\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} \widehat{u}_n(\lambda) h_n^\lambda(x), \tag{17}$$

and obtain

$$\widehat{\Delta_G}(\lambda)\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} -(2n+1)|\lambda| \widehat{u}_n(\lambda) h_n^\lambda(x).$$

Summing up, by writing

$$u(x,y) = \int_{\mathbb{R}} e^{i\lambda y} \left(\sum_{n\in\mathbb{N}} h_n^\lambda(x)\widehat{u}_n(\lambda) \right) d\lambda, \tag{18}$$

we obtain

$$\Delta_{BG} u(x,y) = \int_{\mathbb{R}} |\lambda| e^{i\lambda y} \left(\sum_{n\in\mathbb{N}} -(2n+1) h_n^\lambda(x)\widehat{u}_n(\lambda) \right) d\lambda.$$

Suppose now that the initial datum u_0 is supported only on the Hermite mode $n = \widetilde{n}$ (and, say, on positive Fourier modes $\lambda \geq 0$), that is,

$$u_0(x,y) = \int_0^\infty e^{i\lambda y} h_{\widetilde{n}}^\lambda(x)\widehat{u}_{0,\widetilde{n}}(\lambda) d\lambda, \tag{19}$$

then we realize that a solution is

$$u(x,y,t) = u_0(x, y - (2\widetilde{n}+1)t), \quad \forall t \in \mathbb{R}. \tag{20}$$

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RESTRICTIONS OF FOURIER TRANSFORMS TO QUADRATIC SURFACES AND DECAY OF SOLUTIONS OF WAVE EQUATIONS

ROBERT S. STRICHARTZ

§1. Introduction

Let S be a subset of $\mathbb{R}^n$ and $d\mu$ a positive measure supported on S and of temperate growth at infinity. We consider the following two problems:

Problem A. For which values of p, $1 \leq p < 2$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\hat{f}$ has a well-defined restriction to S in $L^2(d\mu)$ with

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \leq c_p \|f\|_p ? \tag{1.1}$$

Problem B. For which values of q, $2 < q \leq \infty$, is it true that the tempered distribution $Fd\mu$ for each $F \in L^2(d\mu)$ has Fourier transform in $L^q(\mathbb{R}^n)$ with

$$\|(Fd\mu)\hat{}\,\|_q \leq c_q\left(\int |F|^2 d\mu\right)^{1/2} ? \tag{1.2}$$

A lot of contributors: Stein, Fefferman, Tomas, etc.

Problem: Can we restrict Fourier transform of L^p functions to subsets ?

- f in $L^1(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ continuous $\rightarrow$ OK.
- f in $L^2(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ in $L^2(\widehat{\mathbb{R}}^n)$ $\rightarrow$ arbitrary on a zero meas set $\widehat{S}$ of $\widehat{\mathbb{R}}^n$.

- what happens for $1 < p < 2$?
- it depends on the surface!
- if the surface is “flat” we cannot do a lot

First observation

→ The Fourier transform of a L^p function, for any $p > 1$, cannot be restricted to hyperplanes.

- This f belongs to $L^p(\mathbb{R}^n)$, for all $p > 1$

$$f(x) = \frac{e^{-|x'|^2}}{1 + |x_1|} \qquad x = (x_1, x') \in \mathbb{R}^n, \tag{21}$$

- its Fourier transform does not admit a restriction on $\widehat{S} = \{\xi_1 = 0\}$.

$$\widehat{f}(0, \xi') = \int_{\mathbb{R}^n} e^{-ix' \cdot \xi'} \frac{e^{-|x'|^2}}{1 + |x_1|} dx_1 dx'$$

→ what happens for different surfaces?

Tomas-Stein

Theorem (Tomas-Stein, 1975)

Let $\widehat{S}$ be a smooth compact hypersurface in $\widehat{\mathbb{R}}^n$ with non vanishing Gaussian curvature at every point, and let $d\sigma$ be a smooth measure on $\widehat{S}$. Then

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^2(\widehat{S},d\sigma)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}\,.$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$ and every $p \le (2n+2)/(n+3)$,

- A similar result is possible for surfaces with vanishing Gaussian curvature (that are not flat).
- In this case the range of p is smaller depending on the order of tangency of the surface to its tangent space.
- The assumption about compactness of $\widehat{S}$ can be removed by replacing $d\sigma$ with a compactly supported smooth measure.

Dual approach

- The operator R_S is continuous from $L^p(\mathbb{R}^n)$ to $L^q(\widehat{S}, d\sigma)$?

$$R_S f = \mathcal{F}(f)|_{\widehat{S}}$$

$\rightarrow$ not completely settled in its general form

from now on

we focus on the case $q = 2$

- the adjoint operator R_S^* is continuous from $L^2(\widehat{S}, d\sigma)$ to $L^{p'}(\mathbb{R}^n)$?

$$R_S^* g = \mathcal{F}^{-1}(g d\sigma)$$

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(\widehat{S}, d\sigma)} \tag{22}$$

Equivalent to the continuity from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ of the operator

$$R_S^* R_S f = f * \widehat{\sigma} \tag{23}$$

$$\|\mathcal{F}(f)|_{\widehat{S}}\|^2_{L^2(\widehat{S},d\sigma)} = \int (f * \widehat{\sigma}) f dx \leq \|f * \widehat{\sigma}\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

Recall that the Fourier transform of the measure $d\sigma$ is a function given by

$$\widehat{\sigma}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} d\sigma(x) \tag{24}$$

Let S be a smooth compact hypersurfaces with non-zero Gaussian curvature at every point. Then

$$|\widehat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-\frac{n-1}{2}} \tag{25}$$

Let S be a smooth compact hypersurfaces with non-zero Gaussian curvature at every point. Then

$$|\widehat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}} \tag{26}$$

- only with decay one only gets $p \leq \frac{4n}{3n+1}$ (Fefferman, Stein)

$$n = 3, \qquad \widehat{\sigma}(\xi) = 2\frac{\sin(2\pi|x|)}{|x|}$$

- using a dyadic decomposition and real interpolation $p < \frac{2(n+1)}{n+3}$ (Tomas)
- with complex interpolation $p = \frac{2(n+1)}{n+3}$ (Stein)

Given a solution $u(t,x)$ of the classical Schrödinger equation (S) in $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \, , \end{cases}$$

the Fourier transform $\widehat{u}(t,\xi)$ with respect to the spatial variable x satisfies

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi), \qquad \widehat{u}(0,\xi) = \widehat{u_0}(\xi). \tag{27}$$

Solving the corresponding ODE and taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u_0}(\xi) d\xi \, . \tag{28}$$

Another viewpoint

One can also interpreted as the inverse Fourier transform of a data on the paraboloid $\widehat{S}$ in the space of frequencies

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z) d\sigma(z)$$

where $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$, defined as

$$\widehat{S} \stackrel{\text{def}}{=} \left\{ (\alpha, \xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2 \right\}.$$

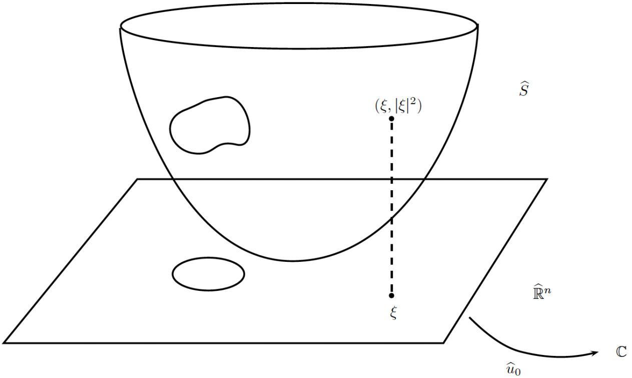
where $y = (t, x)$ and $z = (\alpha, \xi)$

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} = \|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^{n+1})}$$

Geometric interpretation

- Let us endow $\widehat{S}$ with the measure $d\sigma = d\xi$.

→ $d\sigma$ is not the intrinsic surface measure of $\widehat{S}$, which is $d\mu = \sqrt{1 + 2|\xi|}d\xi$.



The Fourier restriction theorem

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\widehat{\mathbb{R}}^{n+1})} \leq C_p \|g\|_{L^2(\widehat{S},d\mu)}, \tag{29}$$

for all $g \in L^2(\widehat{S}, d\mu)$ and all $p' \geq 2(n+2)/n$.

By construction $\|g\|_{L^2(\widehat{S},d\mu)} = \|\widehat{u}_0\|_{L^2(\widehat{\mathbb{R}}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$

$\rightarrow$ apply the result in dimension $n+1$, i.e., in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$

Applying the statement to g related to a initial data u_0 such that $\widehat{u}_0$ is supported on a unit ball

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{30}$$

for all $p' \geq 2(n+2)/n$.

Scaling argument + density of spectrally localized functions in $L^2(\mathbb{R}^n)$, give the result for $p' = 2 + \frac{4}{n}$. and all $u_0 \in L^2(\mathbb{R}^n)$

Some difficulties

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1. Prove a Fourier restriction on the Heisenberg group

- a result of D.Müller $\rightarrow$ specific for the sphere
- what is the sphere? what about paraboloid?

2. We do not exactly need restriction theorems for $\mathbb{H}^d$

- we applied the result to a surface in the space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$

$\rightarrow$ the paraboloid for the Schrödinger eq. (the cone for the wave equation).

- when dealing with equations defined on the Heisenberg group $\mathbb{H}^d$, one is naturally lead to consider surfaces in the space $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$, which is not related to $\mathbb{H}^{d'}$ for some d'.

A function ϕ on $\mathbb{H}^1$ is said to be *radial* if $\phi(x,y,z) = \phi(x^2+y^2,z)$.

Theorem (Bahouri, DB, Gallagher, '21)

Given (p,q) belonging to the admissible set

$$\mathcal{A} = \left\{ (p,q) \in [2,\infty]^2 \;/\; p \leq q \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2} \right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ with radial data satisfies

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{L^2(\mathbb{H}^d)} \right).$$

- restrictive due to $p \leq q$. Indeed $p = q = 2$.
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

The result

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A function ϕ on $\mathbb{H}^1$ is said to be *radial* if $\phi(x, y, z) = \phi(x^2 + y^2, z)$.

Theorem (Bahouri, DB, Gallagher, '21)

Given (p, q) *belonging to the admissible set*

$$\mathcal{A} = \left\{(p, q) \in [2, \infty]^2 \, / \, p \leq q \quad and \quad \frac{2}{q} + \frac{2d}{p} \leq \frac{Q}{2}\right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ *with radial data satisfies*

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{H^\sigma(\mathbb{H}^d)}\right).$$

- $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$ is the loss of derivatives, $\sigma = 0$ forces $p = q$
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

The result (updated)

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A function ϕ on $\mathbb{H}^1$ is said to be *radial* if $\phi(x,y,z) = \phi(x^2+y^2,z)$.

Theorem (DB, Flynn, '24)

Given (p,q) *belonging to the admissible set*

$$\mathcal{A} = \left\{ (p,q) \in [2,\infty]^2 \,/\, p \leq q \quad \text{and} \quad \frac{2}{q} + \frac{2d}{p} \leq \frac{Q}{2} \right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ ~~*with radial data*~~ *satisfies*

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{H^\sigma(\mathbb{H}^d)} \right).$$

- $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$ is the loss of derivatives, $\sigma = 0$ forces $p = q$
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

The Fourier transform on $\mathbb{H}$

It is defined using irreducible unitary representations : for any integrable function u on $\mathbb{H}$ (Kirillov theory)

$$\forall \lambda \in \mathbb{R}^*, \quad \widehat{u}(\lambda) := \int_{\mathbb{H}} u(x) \mathcal{R}_x^\lambda dx\,,$$

with $\mathcal{R}^\lambda$ the group homomorphism between $\mathbb{H}$ and the unitary group $\mathcal{U}(L^2(\mathbb{R}))$ of $L^2(\mathbb{R})$ given for all x in $\mathbb{H}$ and ϕ in $L^2(\mathbb{R})$, by

$$\mathcal{R}_x^\lambda \phi(\theta) := \exp\Big(i\lambda x_3 + i\lambda\theta x_2\Big)\phi(\theta + x_1)\,.$$

Then $\widehat{u}(\lambda)$ is a family of bounded operators on $L^2(\mathbb{R})$, with many properties similar to $\mathbb{R}^d$: $\underbrace{\text{inversion formula}}_{Trace}$, $\underbrace{\text{Fourier-Plancherel identity}}_{Hilbert-Schmidt}$

The sub-Laplacian

$$\Delta_{\mathbb{H}} = X_1^2 + X_2^2$$

There holds

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda) = \widehat{u}(\lambda) \circ P_\lambda\,, \quad \text{with} \quad P_\lambda := -\frac{d^2}{d\theta^2} + \lambda^2\theta^2\,.$$

The spectrum of the **rescaled harmonic oscillator** is

$$\mathrm{Sp}(P_\lambda) = \big\{|\lambda|(2m+1)\,, m \in \mathbb{N}\big\}$$

and the eigenfunctions are the Hermite functions ψ_m^λ. So for all $m \in \mathbb{N}$,

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda)\psi_m^\lambda = E_m(\lambda)\widehat{u}(\lambda)\psi_m^\lambda\,.$$

Set $\widehat{x} := (n, m, \lambda) \in \widehat{\mathbb{H}} = \mathbb{N}^2 \times \mathbb{R}^*$, and

$$\mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda) := (\widehat{u}(\lambda)\psi_m^\lambda | \psi_n^\lambda)_{L^2(\mathbb{R})} = \int_{\mathbb{H}} \mathcal{W}(\widehat{x}, x) u(x) dx$$

where

$$\mathcal{W}(\widehat{x}, x) = \int_{\mathbb{R}^d} e^{2i\lambda\langle y, x'\rangle} H_{n,\lambda}(x + x') H_{m,\lambda}(-x + x')\, dx' .$$

of the (renormalized) Hermite functions $H_{m,\lambda} = |\lambda|^{\frac{1}{4}} H_m(|\lambda|^{\frac{1}{2}} x)$

Then

$$\mathcal{F}_{\mathbb{H}}(-\Delta_{\mathbb{H}} u)(n, m, \lambda) = \underbrace{E_m(\lambda)}_{\text{frequency}} \mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda) .$$

Bahouri, Chemin, Danchin

Inversion and Fourier-Plancherel formulae

$$f(\widehat{x}) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} \mathcal{W}(\widehat{x}, x)\mathcal{F}_{\mathbb{H}} f(\widehat{x})\, d\widehat{x}$$

and

$$(\mathcal{F}_{\mathbb{H}} f | \mathcal{F}_{\mathbb{H}} g)_{L^2(\widetilde{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} (f|g)_{L^2(\mathbb{H}^d)}\,,$$

Action of the Laplacian

$$\mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}} f)(\widehat{x}) = -4|\lambda|(2|m| + d)\mathcal{F}_{\mathbb{H}}(f)(\widehat{x})\,.$$

$\rightarrow$ Radial functions

$$\mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda) = \mathcal{F}_{\mathbb{H}}(f)(n, m, \lambda)\delta_{n,m} = \mathcal{F}_{\mathbb{H}}(f)(|n|, |n|, \lambda)\delta_{n,m}\,.$$

Strichartz estimate in the Heisenberg group

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Let u_0 in $\mathcal{S}(\mathbb{H}^d)$ be radial and consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Taking the partial Fourier transform with respect to the variable w

$$\begin{cases} i\frac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} = \mathcal{F}_{\mathbb{H}} u_0 \,. \end{cases}$$

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m} \,.$$

→ Notice that if we set $|m| = 0$ we see the "transport" part

$$\mathcal{F}_{\mathbb{H}}(u)(t,0,0,\lambda) = e^{4it|\lambda|d}\mathcal{F}_{\mathbb{H}}(u_0)(0,0,\lambda) \,.$$

Applying the inverse Fourier formula

$$u(t,z,s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} \mathcal{W}(\widehat{x},z,s)\, e^{4it|\lambda|(2|m|+d)}\, \mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\, d\widehat{x}\,.$$

Re-expressed as the inverse Fourier transform in $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$ of $\mathcal{F}_{\mathbb{H}}(u_0)\, d\Sigma$,

$$\Sigma \stackrel{\text{def}}{=} \Big\{ (\alpha,\widehat{x}) = \big(\alpha,(n,n,\lambda)\big) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \,/\, \alpha = 4|\lambda|(2|n|+d) \Big\}\,.$$

endow Σ with the measure $d\Sigma$ induced by the projection $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha,\widehat{x})\, d\Sigma(\alpha,\widehat{x}) = \int_{\widehat{\mathbb{H}}^d} \Phi(4|\lambda|(2|m|+d),\widehat{x})\, d\widehat{x},$$

Theorem (Bahouri, DB, Gallagher, '19)

If $1 \le q \le p \le 2$, then for f radial

$$\|\mathcal{F}_{\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d}(f)|_\Sigma\|_{L^2(d\Sigma)} \le C_{p,q}\|f\|_{L^1_s L^q_t L^p_z}\,, \tag{31}$$

Using dual inequality, assuming that $F_{\mathbb{H}} u_0$ is localized in the unit ball

For any $2 \leq p \leq q \leq \infty$

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \|\mathcal{F}_{\mathbb{H}} u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C \|u_0\|_{L^2(\mathbb{H}^d)} \,,$$

- If u_0 is frequency localized in the ball $\mathcal{B}_\Lambda$,

$$u_\Lambda(t,z,s) = u(\Lambda^{-2}t, \Lambda^{-1}z, \Lambda^{-2}s), \qquad u_{0,\Lambda}(z,s) = u_0(\Lambda^{-1}z, \Lambda^{-2}s)$$

- we have

$$\|u_\Lambda\|_{L^\infty_s L^q_t L^p_z} = \Lambda^{\frac{2}{q}+\frac{2d}{p}} \|u\|_{L^\infty_s L^q_t L^p_z}, \qquad \|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{\frac{Q}{2}} \|u_0\|_{L^2(\mathbb{H}^d)} \,,$$

- we infer

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \Lambda^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)} \,.$$

Comments on the proof(s)

- case of radial data for higher dimensional Heisenberg (with Bahouri, Gallagher)

→ explicit expression of the spherical dual measure

- case of non radial data for higher dimensional Heisenberg (with Flynn)

→ analysis of spectral projectors (Müller, Ciatti-Casarino, etc.)

- extension to H-type groups (with Flynn, in progress)

some issues

- corank > 1: dispersion. Link with Strichartz
- $p \leq q$ is technical in this proof. Remove?

H-type groups

Consider a more general Carnot group of step 2 with Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$$

of dimension $2n + m$ where $\dim \mathfrak{h} = 2n$, $\dim \mathfrak{v} = m$ and

$$[\mathfrak{h}, \mathfrak{v}] = \mathfrak{v}, \quad [\mathfrak{h}, \mathfrak{v}] = 0. \tag{32}$$

Fix a scalar product g on $\mathfrak{g}$ such that $\mathfrak{h} \perp \mathfrak{v}$. For any $\mu \in \mathfrak{v}^*$, there is a natural skew symmetric operator $J_\mu : \mathfrak{h} \to \mathfrak{h}$defined by

$$\langle \mu, [X, Y] \rangle = g(X, J_\mu Y). \tag{33}$$

A step 2 Carnot group is *H-type* if for all $\mu \in \mathfrak{v}^*$ we have $J_\mu^2 = -\|\mu\|_g^2 I$, where $I : \mathfrak{h} \to \mathfrak{h}$ is the identity map.

H-type groups

Recall that H-type groups of dimension $2n+m$ have homogeneous dimension $Q = 2n+2m$; we obtain the following result

Theorem (DB, S. Flynn - 2024)

Given $(r,p,q) \in [2,\infty]^3$ *satisfying*

$$p \leq r, q \qquad \text{and} \qquad r \geq 2 + \frac{4}{m-1}$$

$$\frac{2m}{r} + \frac{2}{q} + \frac{2n}{p} \leq \frac{Q}{2}$$

the solution to the Schrödinger equation on an H-type Carnot group

$$\|u\|_{L^r_{\mathfrak{v}} L^q_t L^p_{\mathfrak{h}}} \leq C \|u_0\|_{H^\sigma} \, .$$

where $\sigma = \frac{Q}{2} - \frac{2m}{r} + \frac{2}{q} + \frac{2n}{p}$ *and* $Q = 2n+2m$.

THANKS FOR YOUR ATTENTION

On the surface measure

Recall that for θ being the Fourier transform of a radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d \, d\lambda \,.$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_0^\infty \left(\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2R^{2d+1}}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \theta(n, n, \frac{\pm R^2}{2|n| + d}) \Big)$$

On the surface measure, $R = 1$

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Recall that for θ Fourier transform of radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d \, d\lambda \, .$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_0^\infty \left(\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_1(\widehat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \theta(n, n, \frac{\pm 1}{2|n| + d}) \Big)$$

The result of Müller

- D.Müller [Annals of Math, 1990]: works in terms of spectral decomposition

$$L = \int_0^\infty \lambda dE(\lambda), \qquad \mathcal{P}f = f * G$$

- proves the estimate ("restriction for the sphere"): if $1 \leq p \leq 2$

$$\Big[\sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \Big| \mathcal{F}_{\mathbb{H}}(f)(n, n, \frac{\pm 1}{2|n| + d}) \Big|^2 \Big) \Big]^{\frac{1}{2}} \leq C_p \|f\|_{L^1_s L^p_z}$$

- can be reinterpreted as follows: If $1 \leq p \leq 2$, then for radial f

$$\|\mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C_p \|f\|_{L^1_s L^p_z}\,, \tag{34}$$

→ valid on the full interval: for $p \in [1, 2]$

→ crucial: the anisotropic norm $L^1_s L^p_z$ ($r = 1$ is necessary in vertical)

- false for $p > 2$

Up to a measure zero set on $\hat{\mathbb{H}}^d$

$$\mathbb{S}_{\widehat{\mathbb{H}}^d} = \left\{(n, n, \lambda) \in \widehat{\mathbb{H}}^d \,/\, (2|n| + d)|\lambda| = 1\right\}$$

By definition, the tempered distribution $G = \mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}})$

Lemma

G is the bounded function on $\mathbb{H}^d$ defined by

$$G(z, s) = \frac{2^d}{\pi^{d+1}} \sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \cos\left(\frac{s}{2|n| + d}\right) \mathcal{W}\left(n, n, 1, \frac{z}{\sqrt{2|n| + d}}\right) \tag{35}$$

For the sphere of radius $R^{1/2}$ we have the homogeneity property:

$$G_R(z, s) \stackrel{\text{def}}{=} R^d (G \circ \delta_{\sqrt{R}})(z, s)\,. \tag{36}$$

Measure on the paraboloid

Proceeding as for the restriction theorem on the sphere of $\widehat{\mathbb{H}}^d$, let us first compute

$$G_{\Sigma_{\mathrm{loc}}} \stackrel{\mathrm{def}}{=} \mathcal{F}^{-1}_{\hat{\mathbb{R}}\times\hat{\mathbb{H}}^d}(d\Sigma_{\mathrm{loc}})\,.$$

Lemma

With the above notation, $G_{\Sigma_{\mathrm{loc}}}$ is the bounded function on $\mathbb{R}\times\hat{\mathbb{H}}^d$ defined by

$$G_{\Sigma_{\mathrm{loc}}}(t,w) = 2\pi\int_0^\infty G_\alpha(w)\,e^{-it\,\alpha}\psi(\alpha)\,d\alpha\,, \tag{37}$$

where G_R is the inverse Fourier of the measure of sphere of radius $R^{1/2}$.

This gives for all f in $\mathcal{S}_{\mathrm{rad}}(\mathcal{D})$

$$(R^*_{\Sigma_{\mathrm{loc}}}R_{\Sigma_{\mathrm{loc}}}f)(t,z,s) = \left(\frac{\pi}{2}\right)^d (G_{\Sigma_{\mathrm{loc}}}\star\check{f})(-t,-z,s)\,, \tag{38}$$

Reduction to the estimate on convolution

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Consider the restriction operator

$$R_{\Sigma_{\mathrm{loc}}} f = \mathcal{F}_{\mathbb{R}\times\mathbb{H}^d}(f)_{|\Sigma_{\mathrm{loc}}}$$

Indeed applying the Hölder inequality, we deduce that

$$\begin{aligned}\|R_{\Sigma_{\mathrm{loc}}} f\|^2_{L^2(\Sigma_{\mathrm{loc}})} &\leq \|R^*_{\Sigma_{\mathrm{loc}}} R_{\Sigma_{\mathrm{loc}}} f\|_{L^\infty_s L^{q'}_t L^{p'}_Y} \|f\|_{L^1_s L^q_t L^p_Y} \\ &\leq \|\check{f} \star_{\mathcal{D}} G_{\Sigma_{\mathrm{loc}}}\|_{L^\infty_s L^{q'}_t L^{p'}_Y} \|f\|_{L^1_s L^q_t L^p_Y}\,,\end{aligned}$$

Then as in the Euclidean case, we are reduced to proving that $R^*_{\Sigma_{\mathrm{loc}}} R_{\Sigma_{\mathrm{loc}}}$ is bounded from $L^1_s L^q_t L^p_z$ into $L^\infty_s L^{q'}_t L^{p'}_z$.

Proof for $1 \leq p < 2$ (non endpoint)

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Main lemma

$$\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} \lesssim \left\| \|\mathcal{F}_{\mathbb{R}}(f)(-\alpha,\cdot)\|_{L_z^p L_s^1} \alpha^{d(1-\frac{2}{p'})} \psi(\alpha) \right\|_{L_\alpha^q}$$

- Hölder estimate in α + Hausdorff-Young inequality: for any $a \geq 2$

$$\begin{aligned}\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} &\lesssim \|\mathcal{F}_{\mathbb{R}}(f)\|_{L_\alpha^a L_z^p L_s^1} \|\alpha^{d(1-\frac{2}{p'})} \psi(\alpha)\|_{L_\alpha^b} \\ &\lesssim \|f\|_{L_t^{a'} L_z^p L_s^1} \|\alpha^{d(1-\frac{2}{p'})} \psi(\alpha)\|_{L_\alpha^b(\mathbb{R})},\end{aligned}$$

where a' is the conjugate exponent of a and $\frac{1}{a} + \frac{1}{b} = \frac{1}{q}$.

- Finally for $a' = q$ and Minkowski's inequality, we get for $q' \geq p' > 2$

$$\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} \lesssim \|f\|_{L_s^1 L_t^q L_z^p}$$

→ endpoint $p = 2$: ad hoc argument

The situation for dispersion on general step 2 is different

Theorem (Bahouri-Fermanian-Gallagher 2016)

Let G be a step 2 stratified Lie group with

- *center of dimension p*
- *radical index k.*
- *non-degeneracy assumption (*) holds.*

If $u_0 \in L^1(G)$ is spectrally localized in a ring, then

$$\|u(t,\cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{\frac{k}{2}}(1+|t|^{\frac{p-1}{2}})}\|u_0\|_{L^1(G)}$$

In Heisenberg $k = 0$ and $p = 1$!