Hyponormal quantization

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Contents

Why Markov's transform?

Let μ be a positive measure on $\mathbb{R}$, with all power moments finite. The Cauchy transform

$$F(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{t - z},$$

maps $\Im z > 0$ to $\Im F(z) > 0$.

$\log(1 + F(z))$ exists and $\Im \log F(z) \in (0, \pi)$:

$$\log(1 + F(z)) = \frac{1}{\pi} \int \frac{\phi(t)dt}{t - z}, \quad 0 \leq \phi \leq \pi.$$

Formal exponential transform

The power moments $s_k(g) = \int g(t) t^n dt, \quad k \geq 0$, can be arranged into the generating analytic series (Cauchy transform):

$$\sum_{k=0}^{\infty} \frac{s_k(g)}{z^{k+1}} = -\int \frac{g(t)dt}{t-z}, \quad |z| > 1.$$

Formal exponential transform

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The key property of these rather special generating series is encoded in the formal exponential transform:

$$\exp[-\sum_{k=0}^{\infty} \frac{s_k(g)}{z^{k+1}}] = 1 - \sum_{k=0}^{\infty} \frac{t_k(g)}{z^{k+1}}.$$

Extremal solutions

The moment sequence $(s_k(g))_{k=0}^{\infty}$ of an integrable function with values in $[0,1]$ is characterized by the positive semi-definiteness of the Hankel matrix $(t_{k+\ell}(g))_{k,\ell=0}^{\infty}$.

The measure $g(t)dt$ is determined by finitely many of its moments if and only if there exists an integer d, such that

$$\det[t_{j+\ell}(g)]_{j,\ell=0}^{d} = 0,$$

in which case we already know that g is the sublevel set of a polynomial function, that is a **finite collection of intervals**.

Matrix perturbation

Let A, B be self-adjoint, $d \times d$ complex matrices. Assume

$$B - A = \xi\langle \cdot, \xi\rangle = \xi \otimes \xi.$$

The min-max principle implies:

$$\lambda_1(A) \le \lambda_1(B) \le \lambda_2(A) \le \lambda_2(B) \le \ldots \le \lambda_d(A) \le \lambda_d(B).$$

Denote

$$g = \sum_{j=1}^{n} \chi_{[\lambda_j(A), \lambda_j(B)]}.$$

Perturbation determinant

Then

$$\det(B-z)(A-z)^{-1} = \prod_{j=1}^{d} \frac{\lambda_j(B)-z}{\lambda_j(A)-z} = \exp \int \frac{g(t)dt}{t-z}.$$

On the other hand

$$\det(B-z)(A-z)^{-1} = \det[I + (A-z)^{-1}\xi \otimes \xi] =$$

$$1 + \langle (A-z)^{-1}\xi, \xi \rangle = 1 + \int \frac{d\mu(t)}{t-z},$$

in view of the spectral theorem.

The phase shift

For any polynomial $p \in \mathbb{C}[X]$ one has

$$\mathrm{trace}[p(B) - p(A)] = \int p'(t)g(t)dt.$$

Note that $g(t)$ is any extremal solution to the L-problem of moments on the real line.

Infinite dimension

There exists a *constructive* bijective correspondence between:

1) Linear bounded self-adjoint operators A with a prescribed cyclic vector ξ;

2) Functions $g \in L^1_{comp}(\mathbb{R}, dx)$ with values in $[0, 1]$;

3) Positive measures μ of compact support on the real line.

$$\det(A + \xi \otimes \xi - z)(A - z)^{-1} = 1 + \langle (A - z)^{-1}\xi, \xi \rangle =$$

$$1 + \int \frac{\mu(dx)}{x - z} = \exp \int \frac{g(t)dt}{t - z}, \quad \Im z > 0.$$

$$\mathrm{trace}(f(A + \xi \otimes \xi) - f(A)) = \int f'(t)g(t)dt, \quad f \in \mathcal{C}^{(1)}(\mathbb{R}).$$

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Hyponormal opperator

Is a linear bounded operator T acting on a Hilbert space H subject to ithe commutator inequality

$$[T^*, T] = T^*T - TT^* \geq 0$$

That is, for every vector $x \in H$, on has

$$\langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle,$$

or equivalently

$$\|Tx\| \geq \|T^*x\|, \quad x \in H.$$

Examples

If $\mathrm{S} = \mathrm{N}|_H$ is the restriction of a normal operator to an invariant subspace H, then

$$\|\mathrm{S}x\| = \|\mathrm{N}x\| = \|\mathrm{N}^*x\| \geq \|\mathrm{P}\mathrm{N}^*x\| = \|\mathrm{S}^*x\|, \quad x \in H,$$

where P denotes the orthogonal projection of the larger Hilbert space onto H.

Or a singular integral transform: consider $L^2(I, dx)$, where I is a closed interval on the line. Let $a, b \in L^\infty(I)$, with $a = \overline{a}$, a.e. Obviously the multiplication operator $[\mathrm{X}\phi](x) = x\phi(x)$ is self-adjoint on $L^2(I, dx)$. The operator

$$[\mathrm{Y}\phi](x) = a(x)\phi(x) - \frac{b(x)}{\pi i}\int_I \frac{\overline{b(y)}\phi(y)}{y - x}dy,$$

is well defined as a principal value and bounded on L^2, by the well known continuity of the Hilbert transform.

Then

$$[\mathrm{X}, \mathrm{Y}]\phi(x) = \frac{b(x)}{\pi i} \int_I \overline{b(y)}\phi(y)dy,$$

hence $T = \mathrm{X} + i\mathrm{Y}$ is a hyponormal operator:

$$[T^*, T] = 2i[\mathrm{X}, \mathrm{Y}] \geq 0.$$

Main inequalities

Putnam:

$$\pi\|[T^*, T]\| \leq \mathrm{Area}\ \sigma(T).$$

Berger and Shaw:

$$\mathrm{Trace}[T^*, T] \leq \frac{m(T)}{\pi}\mathrm{Area}\ \sigma(T),$$

where $m(T)$ stands for the *rational multiplicity* of T, that is the minimal number of vectors h_j, $1 \leq j \leq m(T)$, so that $f(T)h_j$ span the whole Hilbert space on which T acts, where f is an arbitrary rational function, analytic in a neighborhood of $\sigma(T)$.

Rank-one self commutator

$$[T^*, T] = \xi \otimes \xi.$$

and T is irreducible, that is the linear span of vectors $T^n T^{*m}\xi$, $n, m \geq 0$ is dense in H.

Then the *multiplicative commutator*

$$(T - z)(T^* - \overline{w})(T - z)^{-1}(T^* - \overline{w})^{-1}$$

is in the determinant class (that is the identity plus a trace-class operator) and

$$\det(T - z)(T^* - \overline{w})(T - z)^{-1}(T^* - \overline{w})^{-1} =$$

$$\det[\mathrm{I} - (\xi \otimes \xi)(T - z)^{-1}(T^* - \overline{w})^{-1}] =$$

$$1 - \langle (T - z)^{-1}(T^* - \overline{w})^{-1}\xi, \xi\rangle =$$

$$1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi\rangle.$$

Pincus Theorem

The integral representation

$$1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle = \exp\left(\frac{-1}{\pi}\int_{\mathbb{C}} \frac{g(\zeta)dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}\right),$$

establishes, for $|z|, |w| >> 1$, a one-to-one correspondence between all irreducible hyponormal operators T with rank-one self-commutator $[T^*, T] = \xi \otimes \xi$ and L^1-classes of Borel measurable functions $g : \mathbb{C} \longrightarrow [0, 1]$ of compact support.

Principal function

The function g is called the *principal function* of the operator T, and it can be regarded as a generalized Fredholm index which is defined even for points of the essential spectrum. Defined whenever $[T^*, T]$ is trace class.

In that case **Helton and Howe Theorem** states:

$$\mathrm{trace}[p(T,T^*), q(T,T^*)] = \frac{1}{\pi}\int_{\mathbb{C}} J(p,q) g \, dA, \quad p, q \in \mathbb{C}[z, \overline{z}],$$

where $J(p,q)$ stands for the Jacobian of the two smooth functions.

Dawn of cyclic cohomology.

The exponential transform

Let $g \in L^1_{comp}(\mathbb{C}, dA)$ have values in $[0, 1]$:

$$E_g(z, \overline{w}) = \exp\left(\frac{-1}{\pi} \int_{\mathbb{C}} \frac{g(\zeta) dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}\right)$$

originally defined for $z, w \notin \mathrm{supp}(g)$ has a series of defining positivity properties encoded in the Hilbert space factorization:

$$E_g(z, w) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi \rangle.$$

It extends separately as a continuous function over the support g, equal to the spectrum of T.

Markov's problem in 2D

The frame is the unit disk $\mathbb{D}$, with test space filled by measurable functions $g : \mathbb{D} \longrightarrow [0, 1]$. We write the power moments in complex coordinates:

$$s_{k\ell}(g) = \int_{\mathbb{D}} z^k \overline{z}^\ell g dA, \quad k, \ell \geq 0,$$

where dA stands for Lebesgue area measure on the disk $\mathbb{D}$.

The formal generating series and its exponential transform are

$$\exp[\frac{-1}{\pi} \sum_{k,\ell=0}^{\infty} \frac{s_{k\ell}(g)}{z^{k+1}\overline{z}^{\ell+1}}] = 1 - \sum_{k,\ell=0}^{\infty} \frac{b_{k\ell}(g)}{z^{k+1}\overline{z}^{\ell+1}}.$$

In particular the matrix $(b_{k\ell}(g))_{k,\ell=0}^{\infty}$ is positive semi-definite.

Extremal solutions

$$\det[b_{k\ell}(g)]_{k,\ell=0}^{d} = 0$$

for some positive integer d, if and only if the original shade function g is the characteristic function of a *quadrature domain* Ω contained in $\mathbb{D}$.

Quadrature domains

A *quadrature domain* is a bounded open set $\Omega \subset \mathbb{C}$ satisfying a Gaussian type quadrature

$$\int_\Omega f(z)dA(z) = c_1 f(a_1) + \ldots + c_d f(a_d),$$

valid for all complex analytic functions f which are integrable on Ω. Above the nodes $a_1, \ldots, a_d$ belong to Ω and the weights $c_1, \ldots, c_d$ are positive. Higher multiplicity nodes, that is derivatives of f, are permitted in such an identity.

A disk is a quadrature domain, in view of Gauss mean value theorem.

A disjoint union of disks is a QD.

The conformal image of a disk by a rational function is also a quadrature domain (such as a cardiodid or a lemniscate).

Algebraic boundary

Any connected quadrature domain is a principal semi-algebraic set, with an irreducible defining polynomial: $\Omega = \{z \in \mathbb{C},\ Q(z, \overline{z}) < 0\}$ (modulo a finite set), where

$$Q(z, \overline{z}) = |P_d(z)|^2 - |P_{d-1}(z)|^2 - |P_{d-2}(z)|^2 - \ldots - |P_1(z)|^2 - |P_0(z)|^2,$$

with $P_j \in \mathbb{C}[z], 0 \leq j \leq d$, and $\deg P_j = j,\ 0 \leq j \leq d$.

Quadrature domains are dense in Hausdorff metric among all bounded open subsets of the complex plane.

Rationality of exp transform

The degenerate situation $\det[b_{k\ell}(g)]_{k,\ell=0}^{d} = 0$ is reflected in the rationality of the exponential transform

$$E_g(z, \overline{w}) = \frac{Q(z, \overline{w})}{P_d(z)\overline{P_d(w)}}, \quad |z|, |w| \to \infty,$$

and vice-versa, provided the degeneracy degree d is chosen minimal.

The nodes $a_1, \ldots, a_d$ of the mechanical quadrature are exactly the zeros of the leading polynomial $P_d(z)$.

Accessible potential, hence effective reconstruction algorithm

The exp transform of the characteristic function $E_G = E_{\chi_G}$ shares the features of a numerically accessible, defining potential:

- $\lim_{z\to\infty} E_G(z,\overline{z}) = 1$,
- $E_G(z,\overline{z})$ is superharmonic and positive on $\mathbb{C}\setminus G$
- $E_G(z,\overline{z}) \sim \mathrm{dist}(z,\partial G), \quad z \to \partial G, \; z \notin G$,
- $E_G(z,\overline{z})$ extends as a real analytic function acros analytic arcs of ∂G.

For instance, in the case of a disk $D(a,r)$ elementary computations yield:

$$E_{D(a,r)}(z,\overline{z}) = 1 - \frac{r^2}{|z-a|^2}, \quad |z-a| > r.$$

Reconstruction of a disk

$$|z-c|^2 \leq M^2, \quad c \in \mathbb{C}, \ M > 0,$$

The initial moments are:

$$a_{00} = \pi M^2,$$

$$a_{01} = \int_{|z-c| \leq M} z dA(z) = \pi M c = \overline{a_{10}},$$

$$a_{11} = \int_{|z-c| \leq M} |z|^2 dA(z) = 2\pi \int_0^M (|c|^2 + r^2) r dr = \pi M^2 |c|^2 + \pi \frac{M^4}{2}.$$

Markov transform

The truncated exponential transforms is:

$$\exp[-\frac{M^2}{z\overline{w}} - \frac{M^2\overline{c}}{z\overline{w}^2} - \frac{M^2 c}{z^2\overline{w}} - \frac{M^2|c|^2 + \frac{M^4}{2}}{z^2\overline{w}^2}] =$$

$$1 - \frac{M^2}{z\overline{w}} - \frac{M^2\overline{c}}{z\overline{w}^2} - \frac{M^2 c}{z^2\overline{w}} - \frac{M^2|c|^2}{z^2\overline{w}^2} + O(\frac{1}{w^3}, \frac{1}{\overline{z}^3}).$$

We infer

$$b_{00} = M^2,\ b_{10} = M^2 c,\ b_{01} = M^2\overline{c},\ b_{11} = M^2|c|^2.$$

Vanishing determinant $b_{00}b_{11} - b_{10}b_{01} = 0$ identifies the monic factor $P(z) = z - c$ as the denominator $P(z)\overline{P(w)}$ of the rational approximant of the full exponential transform. Then

$$(z-c)(\overline{w}-\overline{c})[1 - \frac{M^2}{z\overline{w}} - \frac{M^2\overline{c}}{z\overline{w}^2} - \frac{M^2 c}{z^2\overline{w}} - \frac{M^2|c|^2}{z^2\overline{w}^2}] =$$

$$(z-c)(\overline{w}-\overline{c}) - M^2 + O(\frac{1}{z^2}, \frac{1}{\overline{w}^2}).$$

Conclusion: the generating shape possessing moments $a_{00}, a_{10}, a_{01}, a_{11}$ is necessarily black and white, defined by equation $|z-c|^2 \leq M^2$.

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Harold S. Shapiro

(1928-2021)

Superresolution

Let $\Delta = [-1, 1]^n$ denote the cube in $\mathbb{R}^n$ endowed with Lebesgue measure and fix a degree $d \geq 1$.
Let $p(X) = \sum_{|\beta| \leq d} p_\beta X^\beta$ be a non-constant polynomial and let α be an admissible multi-index with respect to p. Denote by χ the characteristic function of the super-level set $p(x) \geq 0, x \in \Delta$.

Superresolution

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Then

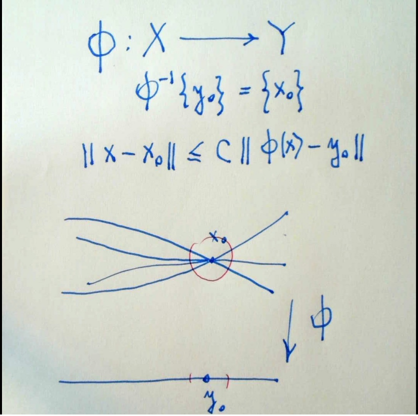
$$\|\chi - g\|_1^{|\alpha|+1} \leq C^{|\alpha|}(1+|\alpha|)|\sum p_\beta(s_\beta(\chi) - s_\beta(g))|$$

for every measurable function g in the ball $\|\chi - g\|_1 \leq \frac{|p_\alpha|^{1/|\alpha|}}{4d} C$, where the constant C depends only on n.

Admissible indices

A multi-index $\alpha \in \mathbb{N}^n$ *admissible with respect to p*, if $p_\alpha \neq 0$ and there exists a permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ of $(1, 2, \ldots, n)$ such that for every β with $p_\beta \neq 0$, either $\alpha_{\sigma(1)} > \beta_{\sigma(1)}$, or there exists an index $j, j \geq 2$, satisfying $\alpha_{\sigma(j)} > \beta_{\sigma(j)}$ and $\alpha_{\sigma(k)} = \beta_{\sigma(k)}$ for $1 \leq k \leq j-1$.

It is easy to see that every polynomial admits at least one admissible multi-index.



ϕ : X ⟶ Y
ϕ⁻¹{y₀} = {x₀}
‖x − x₀‖ ≤ C ‖ϕ(x) − y₀‖
x₀
ϕ
y₀

Ingredient in the proof

Denote $\Delta_r = [-r, r]^n$ for $r > 0$ and

$$V_\delta(p) = \{x \in \mathbb{R}^n,\ |p(x)| < \delta\}.$$

Theorem. (Dieu-2018) Let $p \in \mathbb{R}[x]$ be a polynomial of degree d and let $\alpha \in \mathbb{N}^n$ be an admissible multi-index for p. There is a constant C' depending only on n, such that for every $\delta > 0$ and $r > 0$ one has

$$\mathrm{vol}(V_\delta(p) \cap \Delta_r) \leq C'[\frac{4d}{|p_\alpha|^{1/|\alpha|}}\delta^{1/|\alpha|}r^{n-1} + (\frac{4d}{|p_\alpha|^{1/|\alpha|}}\delta^{1/|\alpha|})^n].$$

General framework

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D. H. Phong, E. M. Stein, and Jacob Sturm. *Multilinear level set operators, oscillatory integral operators, and Newton polyhedra.* Math. Ann., 319(3):573 - 596, 2001.

P. T. Gressman, *Uniform geometric estimates of sublevel sets*, J. Analyse Math. 115(2011), 251- 272.

Second ingredient in the proof

More convexity and duality:

A. S. Lewis. *Superresolution in the Markov moment problem.* J. Math. Anal. Appl., 197(3):774–780, 1996.

Treating exclusively measures on $[0, 1]$.

Equivalent statement

Having the moment space $\mathbb{R}^N$ (containing $s(g) = (s_\alpha(g))_{|\alpha| \leq d}$) endowed with a norm $\| \cdot \|$:

$$\|\chi - g\|_1 \leq K \|s(\chi) - s(g)\|^{\frac{1}{|\alpha|+1}},$$

with a constant K depending on the polynomial p, the admissible multi-index α and n.

2D

Let $\Omega \subset \mathbb{D}$ be a quadrature domain of order d, with characteristic function χ and defining polynomial equation $Q(z, \overline{z}) < 0$, assuming the leading polynomial P_d monic. The highest order term of $Q(z, \overline{z})$ is $|z|^{2d}$, hence the multi-index $(2d, 0)$ is admissible for Q, with coefficient equal to 1.

Theorem. Let $g : \mathbb{D} \longrightarrow [0, 1]$ be a measurable function. Then

$$\|\chi - g\|_{1,\mathbb{D}} \leq e^{1/e} C |[\sum_{j=0}^{d} |\|P_j\|^2_{2,\Omega} - \|P_j\|^2_{2,gdA}|]^{\frac{1}{2d+1}},$$

provided $\|\chi - g\|_1 \leq \frac{C}{4d}$, where C is a universal constant.

Padé type approximation

Let $E(z,\overline{w}) = 1 - \sum_{k,\ell=0}^{\infty} \frac{b_{k\ell}}{z^{k+1}\overline{w}^{\ell+1}}$ be the exponential transform of a measurable function of compact support g, $0 \le g \le 1$, attached to the hyponormal operator T. Fix a positive integer N.

There exists a unique formal series

$$\mathfrak{E}(z,\overline{w}) = 1 - \sum_{k,\ell=0}^{\infty} \frac{c_{k\ell}}{z^{k+1}\overline{w}^{\ell+1}}$$

with the matching property

$$c_{k\ell} = b_{k\ell} \text{ for } (0 \le k \le N-1,\, 0 \le \ell \le N) \text{ or } (0 \le k \le N,\, 0 \le \ell \le N-1)$$

and positivity and rank constraints

$$(c_{k\ell})_{k,\ell=0}^{\infty} \ge 0, \quad \mathrm{rank}(c_{k\ell})_{k,\ell=0}^{\infty} \le \min(N,n)$$

where $n = \mathrm{rank}(b_{k\ell})_{k,\ell=0}^{N}$.

In this case

$$\mathfrak{E}(z,\overline{w}) = E_N(z,\overline{w}) = 1 - \langle (T_N^* - \overline{w})^{-1}\xi, (T_N^* - \overline{z})^{-1}\xi \rangle$$

where T_N the finite central truncation of the operator T to the linear subspace generated by the vectors $\xi, T^*\xi, \ldots, T^{*(N-1)}\xi$.
Moreover,

$$E_N(z,\overline{w}) = \frac{Q_N(z,\overline{w})}{P_N(z)\overline{P_N(w)}},$$

where P_N is the associated orthogonal polynomial, whenever it is unambiguously defined, and the polynomial kernel $Q_N(z,\overline{w})$ is positive semi-definite and has degree at most $N-1$ in each variable.
In addition, $\mathfrak{E}(z,\overline{w}) = E(z,\overline{w})$ as formal series if and only if the function g is the characteristic function of a quadrature domain of order $d \leq N$.

One line formula

If $\Omega = \{z \in \mathbb{C}; Q(z, \bar{z}) < 0\}$ is a quadrature domain, with nodes at the zeros of the polynomial $P(z)$, then

$$\frac{Q(z, \overline{w})}{P(z)\overline{P(w)}} = \exp\left(\frac{-1}{\pi} \int_{Q(\zeta,\bar{\zeta})<0} \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \overline{w})}\right),$$

for $|z|, |w| >> 1$.

Three term relation

The exponential orthogonal polynomials $P_N(z)$ appearing in the approximation scheme satisfy a three term relation if and only if $g = \chi_{\mathcal{E}}$, where $\mathcal{E}$ is an ellipse.

Two point quadrature domains

The quadrature

$$\int_\Omega f dA = \pi r^2 (f(-1) + f(1))$$

valid for all entire functions $f(z)$ has the exponential transform

$$E_\Omega(z, \overline{w}) = \frac{z^2\overline{w}^2 - z^2 - \overline{w}^2 - 2r^2 z\overline{w}}{(z^2-1)(\overline{w}^2-1)}.$$

A double point

The quadrature

$$\int_\Omega f dA = 6\pi f(-1) - 4\pi f'(-1)$$

has a unique representing domain, of exponential transform

$$E_\Omega(z, \overline{w}) = \frac{z^2\overline{w}^2 + 2z\overline{w}^2 + 2z^2\overline{w} + z^2 + \overline{w}^2 - 2z\overline{w}}{(z-1)^2(\overline{w}-1)^2}.$$

The two block-diagonal matrix model

$\Omega \subset \mathbb{C}$ is a quadrature domain with T its hyponormal quantization: $[T^*, T] = \xi\langle\cdot, \xi\rangle$ and principal function equal to the characteristic function of Ω.

In this case the T^*- cyclic subspace $H_0 = \mathrm{span}\{T^{*k}\xi,\ k \geq 0\}$ is finite dimensional. The minimal polynomial of the restriction $D_0 = T^{|}ast|_{H_0}$ coincides with the monic polynomial $P(z)$ of degree d vanishing at the quadrature nodes.

The subspaces

$$H_k = \mathrm{span}\{T^j x,\ 0 \leq j \leq k,\ x \in H_0\}$$

have exact dimension

$$\dim H_k = (k+1)d, \quad k \geq 0.$$

The staircase

The entire space H can be decomposed into an orthogonal direct sum:

$$H = H_0 \oplus [H_1 \ominus H_0] \oplus [H_2 \ominus H_1] \oplus \dots,$$

and correspondingly one can write:

$$T = \begin{pmatrix} D_0 & 0 & 0 & \dots & \dots \\ A_0 & D_1 & 0 & & \dots \\ 0 & A_1 & D_2 & 0 & \dots \\ 0 & 0 & A_2 & D_3 & \dots \\ \vdots & & \ddots & \ddots & \end{pmatrix}.$$

A convenient choice of orthogonal bases in each summand $H_{k+1} \ominus H_k = \mathbb{C}^d$ allows us to assume $A_k > 0$ for all $k \geq 0$.

Recurrence

The commutation relation $[T^*, T] = \xi\langle\cdot, \xi\rangle$ is equivalent to the recurrent system of equations

$$[D_k^*, D_k] + A_k^2 = A_{k-1}^2, \quad k \geq 0; \quad A_{-1} = \xi\langle\cdot, \xi\rangle,$$

$$A_k D_{k+1} = D_k A_k, \quad k \geq 0.$$

Note that $\mathrm{trace}\ A_k^2 = \mathrm{trace}\ A_{k-1}^2, \quad k \geq 0$, hence

$$\mathrm{trace}\ A_k^2 = \|\xi\|^2 = \frac{\mathrm{Area}(\Omega)}{\pi},$$

that is the off diagonal entries in are uniformly bounded in norm.

The boundedness of the entire operator T is equivalent to $\sup_k \|D_k\| < \infty$.

Further applications

Matrix model in Laplacian Growth

Regularity of free boundaries

Spectral analysis: separation of the "cloud" in Thomson's Theorem

Asymptotics of the exponential orthogonal polynomials

Packing quadrature domains

Truncated moment problem in 2D

Björn Gustafsson

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