Algebraic positivity, analytic positivity

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G.H.HARDY, 1940

The problem of the representation of an integer n as the sum of a given number k of integral squares is one of the most celebrated in the history of numbers. Its history may be traced back to Diophantus, but begins effectively with Girard's (or Fermat's) theorem that a prime $4m + 1$ *is the sum of two squares. Almost every artihmetician of note since Fermat has contributed to the solution of the problem, and it has its puzzles for us still.*

Minkowski and Hilbert

Every positive integer is the sum of four squares
LAGRANGE

Academie des Sciences de Paris, 1881: "Study the (number of) decompositions of an integer into five squares"

MINKOWSKI, 1882: *Mémoire sur la théorie des formes quadratiques à coefficients entiers*
wins the Grand Prix (the author is 17 years old!)

" The decomposition of an integer into five squares depends on quadratic forms of four variables, just as the representation of an integer as a sum of three squares depends on quadratic forms in two variables, as shown by Gauss."

Inaugural Dissertation

MINKOWSKI, (Königsberg, 1885) with Hilbert as opponent, published as:

Untersuchungen über quadratische formen, Acta Math. 7(1886), 201-256.

Minkowski's thesis: **"It is not probable that every positive form can be represented as a sum of squares"**

clarified and supported by Hilbert

Über die Darstellung definiter formen als Summen von Formenquadraten, Math. Ann. 32(1888), 342-350

1D

$p(x) \geq 0$ for all $x \in \mathbf{R}$

$$p(x) = q(x)^2 \prod_k [r_k(x)^2 + s_k(x)^2]$$

and

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$

or better

$$|a + ib|^2 |c + id|^2 = |(a + ib)(c + id)|^2,$$

imply

$$p(x) = p_1(x)^2 + p_2(x)^2.$$

2D

Hilbert's construction:

Choose 9 points $a_1, \dots a_9 \in \mathbf{R}^2$ on two cubics, so that for every polynomial Q:

$$\deg Q = 3, \; \& \; Q(a_k) = 0 (1 \le k \le 8) \Rightarrow$$
$$Q(a_9) = 0.$$

Let $P(x, y) \ge 0$ on $(x, y) \in \mathbf{R}^2$,

$\mathrm{degP} = 6$ and

$P(a_k) = 0, \quad 1 \le k \le 8$, but $P(a_9) > 0$.

Then it is impossible to have

$$P = \sum_j Q_j^2.$$

1900, HILBERT's 17-th problem

...the question arises whether every definite form may not be expressed as a quotient of sums of squares of forms...
... it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented."

1909, HILBERT solves Waring's problem

using a cubature formula with rational coefficients for

$$(x_1^2 + \ldots + x_5^2)^m = C \int_{|t|\leq 1} (t_1 x_1 + \ldots + t_5 x_5)^{2m} dt_1 \ldots dt_5.$$

1912 HILBERT publishes the book "*Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen*"

based on six independent, and previously published articles (1904-1910).
Background theme: *spectral analysis requires infinite sums of squares (SOS) decompositions, of possibly infinitely many variables*.
Based on concrete equations of mathematical physics and problems of function theory.

Hilbert-Hahn-Hellinger - spectral theorem

$A = A^*$ linear bounded Hermitian operator

$p \in \mathbf{R}[x], \quad p|_{\mathbf{R}} \geq 0 \quad \Rightarrow p(A) \geq 0.$

Proof. $p = \sum q_k^2$ hence

$$p(A) = \sum q_k(A)^* q_k(A) \geq 0.$$

$$p \mapsto p(A)$$

is a monotonic, bounded $(1 \mapsto I)$ calculus. It can be extended to all bounded measurable functions, and represented by a positive, operator valued, measure:

$$p(A) = \int_{\mathbf{R}} x E_A(dx)$$

Fejer-Riesz factorization

$$p(e^{i\theta}) \geq 0, \quad (\theta \in \mathbf{R}) \Rightarrow p(e^{i\theta}) = \sum_k |q_k(e^{i\theta})|^2$$

and one can choose $q(e^{i\theta}) = \sum_{j=0}^{N} c_j e^{ij\theta}$.

Application:

$$U^{-1} = U^*, \; p(e^{i\theta}) \geq 0 \Rightarrow p(U) \geq 0.$$

Fejer's inequality

$$p(e^{i\theta}) = \sum_{j=-d}^{d} p_j e^{ij\theta} \geq 0 \;\Rightarrow$$

$$|p_1| \leq p_0 \;\cos\frac{\pi}{d+2}.$$

Riesz-Herglotz formula

$p \in \mathbf{C}[z],\ \Re p(z) \geq 0\ (|z| \leq 1)$

can be written:

$$p(z) = i\Im p(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\Re p(e^{i\theta})d\theta.$$

Consequently, denoting $d\mu(e^{i\theta}) = \Re p(e^{i\theta})\frac{d\theta}{2\pi}$:

$$\frac{p(z)+\overline{p(w)}}{1-z\overline{w}} = \int_{|u|=1} \frac{d\mu(u)}{(1-\overline{u}z)(1-u\overline{w})}$$

$$= \langle \frac{1}{1-u\overline{w}}, \frac{1}{1-u\overline{z}} \rangle_{2,\mu}$$

$$\approx \sum_k \frac{c_k}{(1-u_k\overline{w})(1-\overline{u_k}z)},$$

with $c_k > 0$.

Non-negative definite kernel as a continuous SOS, or discrete, but infinite SOS.

Thus, any non-negative harmonic polynomial in the unit disk **D** can be decomposed as:

$$\Re p(z) = (1 - |z|^2) \sum_{k=1}^{\infty} |f_k(z)|^2,$$

with $f_k \in \mathcal{O}(\mathbf{D})$.

Stieltjes, Schur, Nevanlinna

analyticity + SOS (= positivity assumption) has an additional and very useful continued fraction structure. Example:
If $p : \mathbf{D} \longrightarrow \mathbf{D}$, then

$$p(z) = \frac{zr_1(z) + \gamma_0}{1 + \overline{\gamma_0} zr_1(z)}$$

and $\deg r_1 < \deg p$,...

Clarifies classical bounded interpolation problems for analytic functions.

Kolmogorov, 1940

Let X be a set and $K : X \times X \longrightarrow \mathbf{C}$ a positive semi-definite kernel, that is

$$(K(x_i, x_j))_{i,j \in I} \geq 0,$$

as a matrix, for all finite subsets $I \subset X$.
Then there exists a Hilbert space H and a map $k : X \longrightarrow H$ such that

$$K(x, y) = \langle k(x), k(y) \rangle, \quad x, y \in X.$$

$$K(x, y) = \langle \sum_j k_j(x) e_j, \sum_m k_m(y) e_m \rangle$$

$$= \sum_j k_j(x) \overline{k_j(y)}.$$

Real algebra and logic

E. Artin, Über die Zerlegung definiter Funktionen in Quadrate, Abh. math. Sem. Hamburg 5(1926), 100-115.
E. Artin, O. Schreier, Algebraische Konstruktion reeler Körper, Abh. math. Sem. Hamburg 5(1926), 85-99.

Extensions of real fields, and of positive cones (orders).

The following are equivalent:

a). K is a real closed field;
b). K^2 is a positive cone and every odd degree polynomial in $K[X]$ has a root in K;
c). $K \neq K(\sqrt{-1})$ and $K(\sqrt{-1})$ is algebraically closed.

Every ordered field K has a real closure, unique up to a K-isomorphism.
Ideas originating in Hilbert's *Foundations of Geometry*.

Application: *Every non-negative polynomial is a SOS of rational functions.*

SOS in K are *totally positive* elements, i.e. positive w.r. to every ordering of K.

Sturm's theorem

Start with $(f, f') = 1$ and define $f_i, g_i \in \mathbf{R}[X]$ as:

$$f_0 = f, f_1 = f',$$

$$f_0 = g_1 f_1 - f_2, \quad \deg f_2 < \deg f_1,$$

$$\vdots$$

$$f_{m-2} = g_{m-1} f_{m-1} - f_m, \quad \deg f_m = 0.$$

Consider an interval $[a, b] \subset \mathbf{R}$, such that $f_j(a) f_j(b) \neq 0$ for all j. Let $N(x)$ be the number of sign changes in the sequence $f_0(x), f_1(x), \ldots, f_m(x)$. Then

The number of roots of f in $[a, b]$ is $N(a) - N(b)$.

Tarski's elimination theory for real closed fields

To any formula $\phi(X_1, ..., X_n)$ *in the vocabulary* $\{0, 1, +, \cdot, <\}$ *and with variables in a real closed field, one can effectively associate two objects:*
(i) a quantifier free formula $\overline{\phi}(X_1, ..., X_n)$ *in the same vocabulary, and*
(ii) a proof of the equivalence $\phi \equiv \overline{\phi}$ *that uses only the axioms of real closed fields.*

Completeness of elementary algebra and geometry, based on an effective decision method. If valid in one real closed field, then it is valid in all other. Idea later developed by Lefschetz in his transfer principle.

Main question: does the system

$$f(X) = 0,\ g_1(X) > 0, \ldots,\ g_k(X) > 0$$

admit a solution in a real closed field R, based on a criterion which is *rational* in the coefficients of the polynomials $f, g_1, \ldots, g_k$?

Lectures in 1927, announced in *Ann. Soc. Pol. Math.* 9(1931), published in *Fund. Math.* 17(1931); full manuscript of 1940 published by 1948 (RAND); Inst. B. Pascal 1967.

Simplification by Seidenberg: *A new decision method for elementary algebra*, Ann. Math. 60(1954), 365-374.

Applications came late, but were spectacular.

Hörmander's inequality (1955):

For each polynomial $f(X_1, ..., X_n) \in \mathbf{R}[X_1, ..., X_n]$ *there are positive constants* c, r *such that*

$$|f(x)| \geq c \operatorname{dist}(x, V(f))^r, \quad x \in \mathbf{R}^n, |x| \leq 1.$$

Generalized to real analytic functions by Lojasiewicz (1964).

Division of distributions by real analytic functions, i.e. elementary solutions to constant coefficient linear PDE's. Hypoellipticity...

In practice:

A basic semi-alegbraic set

$$\{x \in \mathbf{R}^n;\ f(x) = 0, q_1(x) > 0, ..., q_k(x) > 0\}$$

has semi-algebraic projections.

Root separation

Simplest decision problem:

$$x^2 + 2bx + c > 0, \quad x \in \mathbf{R},$$

is equivalent to:

$$b^2 < c.$$

Then one can complete the squares

$$x^2 + 2bx + c = (x + b)^2 + (c - b^2).$$

See other similar root separation criteria (Routh, Hurwitz, Cohen, Schur, Takagi).

Early references

Krivine *Anneaux préordonnés*, J. Analyse Math. 12(1964), 307-326.

Dubois *A Nullstellensatz for ordered fields*, Ark. Mat. 8(1969), 111-114.

Stengle *A Nullstellensatz and a Positivstellensatz in semialgebraic geometry*, Math. Ann. 207(1974), 87-97.

Certificate of solvability

of a system of equalities and inequalities. A subset $A \subset \mathbf{R}[x]$ generates:

$I(A)$ - an ideal ($\sum a_i p_i$)

$\prod(A)$ - a multiplicative monoid (with 1) ($a_i a_j ... a_m$)

$T(A)$ - a convex subsemiring ($\sum a_i ... a_k p^2$).

Let $F, G, H \subset \mathbf{R}[x]$. The set of $\mathbf{R}^n$ given by

$$\begin{cases} f(x) = 0 \ (f \in F), \\ g(x) \neq 0 \ (g \in G), \\ h(x) \geq 0 \ (h \in H), \end{cases}$$

is empty if and only if there are elements $a \in I(F), b \in \prod(G), c \in T(H)$ such that

$$b^2 + c = a.$$

Example

Assume $p \in \mathbf{R}[x]$ is non-negative on $\mathbf{R}^n$. Then

$$\{x \in \mathbf{R}^n;\ p(x) \neq 0, -p(x) \geq 0\} = \phi.$$

Thus there exists $b \in \prod(p),\ c \in T(-p)$ such that:

$$b^2 + c = 0,$$

that is:

$$p^{2k} + f^2 - pg^2 = 0,$$

$$p = \frac{p^{2k}}{g^2} + \frac{f^2}{g^2}.$$

$f = 0$ on $g = 0$ implies:

$$f^{2k} + SOS \in (g).$$

Back to Minkowski and Hilbert

G. Cassier (1984), K. Schmüdgen (1991),
M.P. (1991-93).

Let $S = \{x \in \mathbf{R}^n; P_i(x) \geq 0, 1 \leq i \leq N\}$, with P_1 strictly negative at infinity, be compact and assume $f > 0$ on S. Then

$$f \in SOS + P_1\ SOS + ... + P_N\ SOS.$$

No denominators, and no terms of the form $P_i...P_k\ SOS$.

Algebraic proof: T. Jacobi, A. Prestel (2001).

Sketch of proof:

Assume the contrary and separate (à la Minkowski) f from $C = SOS + P_1\ SOS + ... + P_N\ SOS$ by a linear functional:

$$L : \mathbf{R}[x] \longrightarrow \mathbf{R},\ \ L(f) < 0 \leq L|_C.$$

$$L(h\overline{g}) = \langle h, g \rangle_L$$

is a positive semi-definite inner product; can be completed to a Hilbert space, and

$$\langle x_j h, g \rangle_L = \langle h, x_j g \rangle_L$$

$x = (x_1, ..., x_n)$ strongly commuting system of symmetric operators, subject to $P_j(x) \geq 0$. Hence x has compact spectrum, hence it is bounded. With spectral representation:

$$\langle F(x)\mathbf{1}, \mathbf{1} \rangle = \int_S F d\mu$$

Modern real algebraic geometry

S. Basu, R. Pollack, M.-F. Roy, *Algorithms in real algebraic geometry*. Berlin Heidelberg New York: Springer 2003.

Bochnak, Jacek; Coste, Michel; Roy, Marie-Françoise. *Real algebraic geometry.* Translated from the 1987 French original. Revised by the authors. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 36. Springer-Verlag, Berlin, 1998. x+430 pp

Prestel, Alexander; Delzell, Charles N. *Positive polynomials. From Hilbert's 17th problem to real algebra.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001. viii+267 pp.

Scheiderer, Claus. *A course in real algebraic geometry*, Graduate Texts in Math. Springer-Verlag, 2024

Good curves

(Scheiderer, 2004)

Let Y be a smooth affine curve over R (real closed field) which is rational. Then every element $f \in R[Y]$ which is non-negative is a sum of squares.

Let Y be a smooth connected affine curve of genus $g \geq 1$ which has only real points at infinity. Then there exists $f \in R[Y], f \geq 0$, which is not a sum of squares.

Enumerative geometry

Hilbert (1888) *Every real homogeneous polynomial* $p(x, y, z)$ *of degree* 4 *is a sum of three squares of quadratic forms.*

Powers, Reznick, Scheiderer, Sottile (2004)
If in addition the plane curve defined by p *is smooth, then there are exactly* 8 *non-equivalent such representations.*

Bridging the gap between SOS and positive polynomials

Theorem. (Bochner, Krein) The harmonic polynomials $\mathcal{H}$ on a (compact) homogeneous space Q have the property that any linear functional $L : \mathcal{H} \longrightarrow \mathbb{C}$ which satisfies

$$L(|h|^2) \geq 0, \quad h \in \mathcal{H},$$

is represented by a positive measure on Q:

$$L(f) = \int f d\mu, \quad f \in C(Q).$$

Polynomial optimization

Let $S \subset \mathbb{R}^d$ be a (basic) semi-algebraic set and let $p \in \mathbb{R}[x]$.

Claim:

$$p^* = \min\{p(x);\ x \in S\}$$

is equivalent to

$$p(x) - p^* \in SOS \pmod{S}.$$

Too nice to be true. How false it can be?

If S is compact, then always true, for $p(x) - p^* + \epsilon,\ \epsilon > 0$. In general, not true.

Great numerical advantage, because, when true, SOS decompositions can be checked with semidefinite programming, that is optimization over the cone of all positive definite matrices.

N.Z.Shor, *Class of global minimum bounds for polynomial functions*, Cybernetics 23(1987), 731-734.
Y.E.Nesterov, A. Nemirovski, *Interior point methods in convex programming*, SIAM, 1994.
P. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*, PhD Thesis, 2000.

Lasserre's relaxation method

Instead of minimizing

$$p(x), \quad x \in S,$$

consider

$$\min \int_S p(x) d\mu(x),$$

over all probability measures μ supported on S.
Use the moments

$$y_\alpha = \int_S x^\alpha d\mu(x), \quad \alpha \in \mathbf{N}^d,$$

as auxiliary variables.

Always convergent, and it allows an optimal asymptotic/error analysis

Example

Let $p(x) = \sum_\alpha c_\alpha x^\alpha$ and $q(x) = \sum_\gamma a_\gamma x^\gamma$.

$$\min_{q(x)\geq 0} p(x) = \min_y \sum c_\alpha y_\alpha$$

subject to the PSD conditions

$$(y_{\alpha+\beta})_{\alpha,\beta\in\mathbb{N}^d} \succeq 0$$

and

$$(\sum_\gamma a_\gamma y_{\alpha+\beta+\gamma})_{\alpha,\beta\in\mathbb{N}^d} \succeq 0.$$

Applications: 1. Lyapunov stability

$$\frac{dx}{dt} = f(x), \quad x(t) \in \mathbb{R}^d$$

with f polynomial. If there exists $V(x)$ so that

$$V(x) > 0, \quad (\frac{\partial V}{\partial x})^T f(x) < 0,$$

then the system is asymptotically stable.
Relaxation: $V(x)$ polynomial and the two inequalities assured by SOS.

$$x' = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3,$$
$$y' = 3x - y,$$

is stable, with a SOS Lyapunov function of degree 6.

2. Chebyshev-Markov type inequalities

For $S \subset \mathbb{R}$ an interval and X a scalar random variable with unknown distribution, but known moments $\mu_0, \ldots, \mu_n$, bound $P(X \in S)$.

Reduces to

$$\min \sum_{k=0}^{n} c_k \mu_k$$

subject to

$$\sum_{k=0}^{n} c_k x^k \geq 1 \quad x \in S,$$

and

$$\sum_{k=0}^{n} c_k x^k \geq 0 \quad x \in \mathbb{R}.$$

Indeed

$$P(X \in S) = \int_S dP(x) \leq \int_{\mathbb{R}} p(x) dP(x),$$

where $p(x) = \sum_{k=0}^{n} c_k x^k$.

3. Elementary geometry

Let ABC be a triangle with sides a, b, c and area A. Then

$$(4A)^6 \geq 27(a^2+b^2-c^2)^2(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2$$

provided the triangle is acute (*Ono's inequality*).

Proof based on SOS decompositions:

Let $t_1 = a^2+b^2-c^2, t_2 = b^2+c^2-a^2, t_3 = c^2+a^2-b^2$.
The triangle is acute if $t_1, t_2, t_3 \geq 0$.

On the other hand (Heron):

$$A^2 = s^2(s-a)(s-b)(s-c), \quad s = \frac{a+b+c}{2}.$$

Consider

$$P(x,y,z) = (x^4 + x^2y^2 - 2y^4 - 2x^2z^2 + y^2z^2 + z^4)^2 +$$
$$15(x-z)^2(x+z)^2(z^2 - x^2 - y^2)^2.$$

Then

$$(4A)^6 - 27t_1^2t_2^2t_3^2 = P(a,b,c)t_1t_2 + P(b,c,a)t_2t_3 + P(c,a,b)t_3t_1 \geq 0.$$

4. The volume of semi-algebraic sets

Let λ be Lebesgue measure on the cube $C = [-1, 1]^d$ and let $S \subset$ be a basic semi-algebraic subset of C. We want to compute, or only estimate, $\lambda(C)$.
Relaxation method:

$$\sup_{\mu,\nu} \int_S d\nu$$

with the constraints

$$\mu \in M_+(C), \nu \in M_+(S), \int_S x^\alpha d\nu + \int_C x^\alpha d\mu = \int_C x^\alpha d\lambda.$$

5. Spectral estimates of PDO with polynomial coefficients

Find lower bounds of the spectrum of $P(x, D)$ for specific boundary conditions, in terms of SOS factorizations of the form

$$Q(x, D)^*(P(x, D) - \lambda)Q(x, D) = \sum_j F_j(x, D)^* F_j(x, D).$$

Applied to $-\Delta + V(x)$ with $V(x)$ rational (Handy, Bertsimas, Caramanis).

Weyl algebra computations

Generators

$$[p_k, q_j] = -\delta_{kj} i, \quad [p_k, p_j] = [q_k, q_j] = 0$$

and representation

$$\Phi(p_k)f = -i\frac{\partial f}{\partial x_k}, \quad \Phi(q_k)f = x_k f, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Schmüdgen's theorem: If an element $f = P(p_k, q_j)$ in Weyl algebra has even degree and principal part f_{2m}, so that $\Phi(f_{2m}) \geq \epsilon > 0$, then there is a universal "denominator" of the form $\delta = \prod_\ell (-\Delta + \|x\|^2 + \ell)$ such that

$$\delta f \delta = \sum_j g_j^* g_j.$$

The complex twist

Sums of squares

$$\Sigma^2 = \{f^2, f \in \mathbb{R}[x]\} \subset \mathbb{R}[x]$$

Sums of hermitian squares

$$\Sigma_h^2 = \{|h(z)|^2,\ h \in \mathbb{C}[z]\} \subset \mathbb{C}[z, \overline{z}].$$

are obviously universal positive polynomials.

6. Quillen theorem

[mid 1960]
Every positive polynomial on the sphere $S^{2d-1} \subset \mathbb{C}^d$ *is a sum of hermitian squares along* S^{2d-1}*.*

Proof by quantization.

Bargmann-Fock space computations

Let $F(z,\bar{z})$ be a bi-homogeneous form of degree (m,m) with

$$F(z,\bar{z}) > 0, \quad z \neq 0.$$

Then the Toeplitz operator $T_F = F(z, \frac{\partial}{\partial z})$ on Fock space $L^2_a(\mathbb{C}^d, e^{-\|z\|^2} dvol(z))$ is elliptic and leaves invariant the polynomial degree filtration

$$T_F : \mathbb{C}_n[z] \longrightarrow \mathbb{C}_n[z], \quad n \geq 1.$$

By elliptic regularity T_F is Fredholm, hence there exists $N \geq 1$, such that

$$T_{\|z\|^{2N}F} \geq 0$$

and so will be its symbol, as a form

$$\|z\|^{2N}F(z,\bar{z}) \in \Sigma_h^2.$$

Application: spherical isometries

According to Quillen's theorem, if $p(z, \overline{z}) > 0$ on S^{2d-1}, then

$$p(z, \overline{z}) \in \Sigma_h^2 + (1 - \|z\|^2)\Sigma_h^2.$$

If $T = (T_1, , T_d)$ is a commuting tuple satisfying $T_1^* T_1 + \dots T_d^* T_d = I$, then

$$p(T, T^*) \geq 0.$$

Hence there exists a spectral measure E, on a larger Hilbert space, so that

$$p(T, T^*) = \int_{S^{2d-1}} p(\lambda, \overline{\lambda}) E(d\lambda).$$

That is T is *subnormal*.

Proved differently by Athavale in the 1980-ies.

7. Quillen property of real algebraic varieties

$z = x + iy \in \mathbb{R}^d$

$I \subset \mathbb{C}[z, \overline{z}]$ real ideal with compact zero set $V_{\mathbb{R}}(I) \subset \mathbb{C}^d = \mathbb{R}^{2d}$.

(Q) Every $f \in \mathbb{C}[z, \overline{z}]$ satisfying $f|_{V_{\mathbb{R}}(I)} > 0$ belongs to $\Sigma_h^2 + I$.

(S) A commutative tuple of bounded operators T annihilated by I is subnormal.

(Sf) A commutative tuple of finite matrices N annihilated by I is normal.

(D) For every two points $(\alpha, \overline{\alpha}), (\beta, \overline{\beta}) \in V_{\mathbb{R}}(I)$, the mixed coordinate points $(\alpha, \overline{\beta}), (\beta, \overline{\alpha})$ also belong to $V(I)$ (and similar higher multiplicity conditions).

Main result

[2012] P-Scheiderer:

(Q) is equivalent to: *there exist* $C > 0$ *such that*

$$C - \|z\|^2 \in \Sigma_h^2 + I.$$

and the implications

$(Q) \Rightarrow (S) \Rightarrow (Sf) \equiv (D)$

hold

Pseudoconvex boundaries

There are strictly pseudo convex domains in $\mathbb{C}^2$ with real algebraic boundary and which do not satisfy Quillen property.

Example

$$|z_1(z_1^2 - 1)|^2 + |z_2|^2 < c$$

does not fulfill condition (D).

Answers an open question raised by Catlin and D'Angelo (in the 1990-ies).

Hermitian complexity

of a real ideal $I \subset \mathbb{C}[z, \bar{z}]$ is the maximum number of points (with multiplicities) satisfying condition (D).

There are ideals with prescribed hermitian complexity.

D'Angelo-P [2011] *If an ideal I has infinite hermitian complexity, then $V_{\mathbb{R}}(I)$ contains an analytic disk.*

8. Stability of delay systems

The stability of a delay system

$$x'(t) = \sum_{k=0}^{m} A_k x(t - \gamma_k \cdot \tau)$$

is governed by the hyperbolicity of the characteristic function

$$f_\tau(s, A, \tau) = \det(sI - A_0 - \sum_{k=1}^{m} A_k e^{-s\gamma_k \cdot \tau})$$

Reduction on the torus and hyperbolic positivity

Necessary condition:

$$f_z(s, A, z) = \det(sI - A_0 - \sum_{k=1}^{m} A_k z^{\gamma_k}) \neq 0 \quad \text{for} \quad z \in \mathbb{T}^d.$$

Lemma. A polynomial $q(z) \neq 0$ on $\mathbb{T}^d$ iff there are $p_j, r_k \in \mathbb{C}[z]$ such that

$$1 + |p_1(z)|^2 + \ldots + |p_N(z)|^2 = |q(z)|^2(|r_1(z)|^2 + \ldots + |r_N(z)|^2).$$

9. Construction of tight wavelet frames

Refinable vector valued, compactly supported function
$\phi : \mathbb{R}^d \longrightarrow \mathbb{R}^r$ satisfies:

$$\phi(x) = \sum_{|\alpha| \leq N} p(\alpha) U_M T_\alpha \phi(x),$$

where $\alpha \in^d$, $p(\alpha) \in M_{(r,r)}(\mathbb{R})$ and $M \in M_{(d,d)}()$ has spectrum $\sigma(M) \cap \overline{\mathbb{D}} = \emptyset$ and $|\det M| = m$.
Fourier's transform $\hat{\phi}(\omega) = \int \phi(x) e^{-i\omega \cdot x} dx$ satisfies

$$\hat{\phi}(M^T \omega) = P(e^{-i\omega}) \hat{\phi}(\omega), \quad e^{-i\omega} = (e^{-i\omega_1}, ..., e^{-i\omega_d}) \in \mathbb{T}^d.$$

The symbol of refinable functions

$$P(e^{-i\omega}) = \frac{1}{m}\sum_{\alpha} p(\alpha)e^{-i\alpha\cdot\omega}$$

is a matrix valued trigonometric polynomial, lives on a torus, and its algebraic behavior encodes essential properties of the original system of functions, or an extension of it. Cf. A. Ron, Z. Shen (1997), Daubechies, Han, Ron, Shen (2006).

Notation: $P^{\sigma}(e^{-i\omega}) = P(e^{-i(\omega+2\pi\sigma)})$, where $\sigma \in G = 2\pi(M^T)^{-1d}/2\pi^d$ is the symmetry group.

The Unitary Extension Principle

Let $P(e^{-i\omega}) \in M_{(r,r)}(\mathbb{C})$ satisfy $P(1, ..., 1) = I$. If there are polynomials Q_j such that

$$\delta_{\sigma,\tau} - P^{\sigma *} P^{\tau} = \sum_j Q_j^{\sigma *} Q_j^{\tau},$$

then the original family of functions ϕ can be extended to a tight wavelet frame, with symbol $(P, Q_1, ..., Q_N)$.

The scalar case

Theorem.(Charina, P., Scheiderer, Stöckler) If $p \in \mathbb{C}[\mathbb{T}^2]$ satisfies $p(1,1) = 1$ and $\sum_{\sigma \in G} |p^\sigma|^2 \leq 1$, then there are polynomials $q_1, \ldots, q_N \in \mathbb{C}[\mathbb{T}^2]$ such that

$$\delta_{\sigma,\tau} - p^{\sigma *} p^\tau = \sum_j q_j^{\sigma *} q_j^\tau .$$

The same holds in arbitrary dimension, provided the Hessian of $1 - |p^\sigma|^2$ is positive definite at every of its zeros.

The matrix valued, multivariate case

Lemma. Let $F \in M_{(r,r)}(\mathbb{R}[T])$ satisfy $F \geq 0$. If the set

$$\Delta = \{\xi \in T; \det F(\xi) = 0\}$$

is finite, and $\mathrm{rank} F(\xi) = r - 1$ for all $\xi \in \Delta$, then:
(a) If $\dim T \leq 2$, then there exist $G_1, ..., G_N \in M_{(r,r)}(\mathbb{R}[T])$, such that

$$F = G_1^* G_1 + ... + G_N^* G_N.$$

(b) In arbitrary dimension, the same holds true if $\mathrm{Hess}_\xi F > 0$ for all $\xi \in \Delta$.

A. Tarski: *A decision method for elementary algebra and geometry*, Univ. California Press, Berkeley, 1951.

"By a decision method for a class K of sentences (or other expressions) is meant a method by means of which, given any sentence Θ, one can always decide in a finite number of steps whether Θ is in K...
The importance of the decision problem for the whole of mathematics (and for various special mathematical theories) was stressed by Hilbert, who considered this as the main task of a new field of mathematical research for which he suggested the term "metamathematics". "

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Software:

GloptiPoly www.laas.fr/~lasserre
SOSTOOLS www.cds.caltech.edu