Norms on Harish-Chandra modules

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Based on a joint work with Bernstein, Krötz, Kuit,Sayag.

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- What is Harish-Chandra module? Why do we care?
  - Representation theory: Understand groups through their "actions"
  - If G acts on a set X, G acts on functions on X by

$$\pi(g)f(x) = f(g^{-1} \cdot x).$$

The main difficulty is to decide which space of functions to consider.

Motivation for Harish-Chandra module

- Consider the group

$$SU(1,1) := \{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}$$

- $SU(1,1)$ acts on S^1 by fractional linear transformations.
$\rightsquigarrow$ Representation of $SU(1,1)$ in $E = C(S^1)$, $C^\infty(S^1)$, $L^2(S^1)$ and $C^{-\infty}(S^1)$ etc. (Action: $g \cdot f(x) = f(g^{-1} \cdot x)$)

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- (Harish-Chandra) "**algebraic skeleton**" V = functions on S^1 with finite Fourier expansion — trigonometric polynomials $= E^{K-\text{finite}}$.

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- (Harish-Chandra) "**algebraic skeleton**" $V =$ functions on S^1 with finite Fourier expansion — trigonometric polynomials $= E^{K-\text{finite}}$.
- $E^{K-\text{finite}}$ is not closed under the action of $SU(1,1)$. However, it is closed under the action of both $Lie(SU(1,1))$ and K.
- This leads to Harish-Chandra's concept of "Infinitesimal equivalence"

Harish-Chandra's Technology!

Admissible representations (π, E)

$E^{K-\text{finite}} = \bigoplus_{\tau \in \widehat{K}} E[\tau]$
and $dim E[\tau] < \infty$.
$E^{K-\text{finite}} \subset E^{\infty} \rightsquigarrow \mathfrak{g}$
acts on $E^{K-\text{finite}}$.

- (Harish-Chandra) (π, E) irreducible and unitary $\Longrightarrow$ admissibility
- "Infinitesimal equivalence" of E and F means **algebraic equivalence** of $E^{K-\text{finite}}$ and $F^{K-\text{finite}}$.

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- $(\mathfrak{g}, K)$**-module:** By a $(\mathfrak{g}, K)$ module V we understand a module for $\mathfrak{g}$ and K such that
  1. The derived action of K coincides with the action of $\mathfrak{g}$ restricted to $\mathfrak{k} := Lie(K)$.
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  2. V is finitely generated as $U(\mathfrak{g})$-module. ($\rightsquigarrow$ Countable dimension)
- (Harish-Chandra) If (π, E) is an irreducibale unitary representation of G, then $E^{K-\text{finite}}$ is a H-C module.

Globalization questions

- Given a Harish-Chandra module V, a complete locally convex topological vector space E is called a *globalization of V* provided that E supports a G-representation such that $E^{K-\text{finite}} \simeq_{(\mathfrak{g},K)} V$.

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- **(Example)** X- Compact homogeneous space for G, e.g., $X = G/P$ with P minimal parabolic. Then $V := L^2(X)^K \cap L^2(X)^\infty$ –K-finite smooth vectors in the right regular representation of G on $L^2(X)$—Harish-Chandra module. Then $L^2(X)$ is a Hilbert globalization. Also, $C^\omega(X), C^\infty(X)$ with their respective topologies, analytic and smooth globalizations of V.

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- (Schmid)[1] An interesting example is the minimal globalization $V_{\min} = V^\omega$ where V^ω denotes the analytic vectors . The minimal globalization is an instance of a globalization E which is an inductive limit of Banach spaces.

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- (Thanks to Casselman's subrepresentation theorem)

$$V \hookrightarrow I := (\text{Ind}_P^G \sigma)^{K-\text{finite}}$$

  Here $P \subset G$ is a minimal parabolic and σ is a finite dimensional representation of P.
- As I admits many G-continuous norms, for example L^p-norms on $K/K \cap P$ of σ-valued functions, we conclude that every Harish-Chandra module admits G-continuous norms as well.

- **Dual Harish-Chandra module:**

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- Given a G-continuous norm p on V, the dual norm $\widetilde{p}$ defined by

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- p, q – G-continuous norms on V. Then
  1. $\widetilde{\widetilde{p}} = p$.
  2. $p \leq q \iff \widetilde{q} \leq \widetilde{p}$.

- **Matrix coefficients:**
  - p-G-continuous $\rightsquigarrow V_p$ and $\pi_p(g)v$–action of G on V_p.

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  - The matrix-coefficient attached to v and $\widetilde{v}$:

$$m_{v,\widetilde{v}}(g) := \widetilde{v}(g \cdot v).$$

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- **Growth of a representation:** Often modeled by Weights—a locally bounded positive function $w : G \to \mathbb{R}_{>0}$ which is submultiplicative, i.e.

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- Aim: **Structure** the set $\mathrm{Norm}(V, w)$.

  {Banach globalizations with fixed growth rate}

- (G-invariant norms) $w = \mathbf{1}$, in this case $\mathrm{Norm}(V) := \mathrm{Norm}(V, \mathbf{1})$ consists of isometric norms, i.e. norms for which $p(g \cdot v) = p(v)$ for all $g \in G$ and $v \in V$.

Some interesting examples

- **Tempered representations** (Cowling[3], Kunze-Stein)
  - G- semi-simple Lie group with finite center. Let $\pi \in \widehat{G}$ be tempered and V the corresponding Harish-Chandra module. Fix a unitary norm q on V.

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  - The Kunze-Stein phenomenon:

$$p^r(v) \lesssim q(v) \qquad (r > 2)$$

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$$m_{v,\eta}(\Gamma g) := \eta(g \cdot v) \qquad (v \in V, g \in G).$$

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  - Clearly, $q \leq \sqrt{\mathrm{Vol}(X)}p_{\mathrm{aut}}$. Also as shown by BR

$$p_{\mathrm{aut}} \lesssim q_s \iff s > \frac{1}{2}$$

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Structuring Norm(V, w)

There is a natural equivalence relation on Norm(V, w) : For $p, q \in \text{Norm}(V, w)$, we say that $p \sim q$ iff $p \lesssim q$, and $q \lesssim p$. This leads to

$$\mathfrak{Norm}(V, w) := \text{Norm}(V, w)/\sim .$$

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Proposition

If Norm$(V, w) \neq \emptyset$. *Then* $\mathfrak{Norm}(V, w)$ *has a unique minimal element* $[p^w_{\min}]$, *and a unique maximal element* $[p^w_{\max}]$.

Construction of minimal and maximal norms

- Special case: Fix a cyclic vector $\widetilde{v} \in \widetilde{V}$.
  - A representative for the equivalence class of minimal norm:

$$p^{w}_{\min}(v) := \sup_{g \in G} \frac{|m_{v,\widetilde{v}}(g)|}{w(g)} \qquad (v \in V).$$

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  - Also, from $p^w_{\min} \lesssim p$ we obtain that orbit maps $G \to V_{p^w_{\min}}, \; g \mapsto g \cdot v$ are continuous for all $v \in V$.
  - For the maximal norm, dualize the construction!

Sobolev norms and "smooth" globalization

• **Standard Sobolev norm** Fix a norm p on V. Now a fixed basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ we define for every $k \in \mathbb{N}_0$ a norm p_k^{st} by

$$p_k^{\mathrm{st}}(v) := \left(\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq k}} p(X_1^{\alpha_1} \ldots X_n^{\alpha_n} v)^2 \right)^{\frac{1}{2}} .$$

- Different choices of a basis for $\mathfrak{g}$ lead to equivalent norms.
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Theorem 1 (Casselman-Wallach)

For any pair of G-continuous norms p, q on a Harish-Chandra module V there exists a $k \in \mathbb{N}$ such that $p \lesssim q_k^{\mathrm{st}}$.

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Theorem 2 (Casselman-Wallach)

For any pair of G-continuous norms p, q on a Harish-Chandra module V there exists a $k \in \mathbb{N}$ such that $p \lesssim q_k^{\mathrm{st}}$.

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- The identity map $V \to V$ extends to a G-isomorphism of Fréchet spaces $V_p^\infty \to V_q^\infty$. Hence there is up to isomorphism only one SF-globalization of V. This globalization is denoted by V^∞.

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Proposition

Given a G-continuous norm p on a Harish-Chandra module, a vector $v \in V_p$ is smooth if and only if it is K-smooth.

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V–Harish-Chandra module , p a G-continuous norm on V.

- Take $\Delta_K \in U(\mathfrak{k})$ –Laplace element. Consider for any $s \in \mathbb{R}$,

$$D_s := (1 + \Delta_K)^{\frac{s}{2}}.$$

- This acts as a scalar on the K types, i.e.,

$$D_s|_{V[\tau]} = C_\tau^{\frac{s}{2}} \cdot \mathrm{id}_{V[\tau]} \qquad (\tau \in \widehat{K}).$$

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Example

Take $G = SL_2(\mathbb{R})$, and $K = SO(2)$. Then $\widehat{K} \simeq \mathbb{Z}$. For $v = \sum_{n\in\mathbb{Z}} v_n$,

$$p_s(v) = \sum_{n\in\mathbb{Z}} (1 + |n|)^s v_n.$$

p-G-continuous norm on V, and s^{th} Sobolev norm $p_s := p(D_s \cdot)$

- Properties:
  - $p \leq q \Rightarrow p_s \leq q_s$
  - $(p_s)_t = p_{s+t} \qquad (s, t \in \mathbb{R})$
  - (Duality) $\widetilde{(p_s)} = \widetilde{p}_{-s} \qquad (s \in \mathbb{R})$

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  - (Duality) $\widetilde{(p_s)} = \widetilde{p}_{-s} \qquad (s \in \mathbb{R})$
  - (Issue-1) p_s may not be G-continuous. However,

Lemma

For any $k \in \mathbb{N}$ there exists an $s \geq 0$ such that $p_k^{\mathrm{st}} \lesssim p_s$, and vice versa.

- (Issue-2) It is not clear that $(p_s)_{s \geq 0}$ is monotonous. Well, if p is K-Hermitian then, it is certainly is, i.e., $p \lesssim p_s$ holds for $s \geq 0$.

A new invariant

$\bullet$**Sobolev "distance" on** $\mathrm{Norm}(V, w)$
Given $[p], [q] \in \mathfrak{Norm}(V, w)$ we set

$$d^w_{\to}([p],[q]) = \inf\{s \geq 0 \mid p \lesssim q_s\}$$

and define

$$d^w([p],[q]) = \max\{d_{\to}([p],[q]), d_{\to}([q],[p])\}\,.$$

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Lemma

$(\mathfrak{Norm}(V, w), d)$ *is pseudo-metric space.*

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Lemma

$(\mathfrak{Norm}(V, w), d)$ is pseudo-metric space.

• **Sobolev-w-gap**

$$s(V, w) = d([p^w_{\mathrm{min}}], [p^w_{\mathrm{max}}])$$

In other words,

$$s(V, w) := \inf\{s \geq 0 \mid p^w_{\max} \lesssim p^w_{\min,s}\}.$$

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- What if one uses a different family of Sobolev norms? —For example, the standard family $\{p_k^{st}\}$ $\rightsquigarrow$ standard Sobolev gap

$$s^{\mathrm{st}}(V, w) = \min\{k \in \mathbb{N}_0 \mid p^w_{\max} \lesssim (p^w_{\min})^{\mathrm{st}}_k\}$$

Note that the $s^{\mathrm{st}}(V, w)$ is a more coarse invariant of V than the Sobolev gap $s(V, w)$. The sandwiching of Sobolev norms yields universal constants $c, C > 0$ such that

$$cs^{\mathrm{st}}(V, w) \leq s(V, w) \leq Cs^{\mathrm{st}}(V, w)$$

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- *Duality:* $s(V, w) = s(\widetilde{V}, w^\sharp)$. Here $w^\sharp(g) = w(g^{-1})$. In particular, $s(V, w) = s(V, w^\sharp)$ if V is self-dual, i.e. $V \simeq \widetilde{V}$.
- *Monotonicity:* Let w_1, w_2 be two weights with $w_1 \leq w_2$. Then $s(V, w_1) \leq s(V, w_2)$.

• **Infimum construction:** For a family $(q_\alpha)_{\alpha\in\mathcal{A}}$ of seminorms on a vector space E one can define the seminorm $\inf_{\alpha\in\mathcal{A}} q_\alpha$ of the family by

$$\inf_{\alpha\in\mathcal{A}} q_\alpha(v) := \inf_{v=\sum_{\alpha\in\mathcal{A}} v_\alpha} \sum_{\alpha\in\mathcal{A}} q_\alpha(v_\alpha) \qquad (v \in E).$$

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- p^G is the largest semi-norm on V^∞ with $p^G \leq w(g^{-1})p(g\cdot)$.
- If there exists a $q \in \mathrm{Norm}(V,w)$ such that $q \leq p$, then $q \leq p^G$ and p^G is a norm. If in addition there exists a G-continuous norm r on V such that $p \leq r$, then $p^G \in \mathrm{Norm}(V,w)$.
- Let $p \in \mathrm{Norm}(V,w)$. If $s \geq 0$ is such that $p \leq p_s$, then

$$p_s^G := (p_s)^G \in \mathrm{Norm}(V,w).$$

Proposition (Stabilization property)

There exists a $S > 0$ so that

$$[p_{\max}] = [p_s^G] \qquad (s > S).$$

In particular, if p is monotonous, then $S = s(V, w)$.

- **Visualize:**
  - $0 \leq s_1 \leq s_2 \leq \ldots$ ascending chain
  - $p \leq p_{s_1} \leq p_{s_2} \leq \ldots$
  - ascending chain in $\mathrm{Norm}(V, w)$: $p^G \leq p^G_{s_1} \leq p^G_{s_2} \leq \ldots$ –becomes stationary when taking equivalence classes!

G-invariant norms

Consider the case where $w = \mathbf{1}$, in which case $\mathrm{Norm}(V) := \mathrm{Norm}(V, \mathbf{1})$ consists of isometric norms, i.e. norms for which $p(g \cdot v) = p(v)$ for all $g \in G$ and $v \in V$. As before we write p_{min} and p_{max} for representatives of the minimal and maximal element in $\mathfrak{Norm}(V)$, respectively.

Theorem 3

Assume that V is unitarizable and let q be a unitary norm. Then, in the pseudometric space $(\mathfrak{Norm}(V), d)$:

$$d([q],[p_{\min}]) = d([q],[p_{\max}])$$

and in particular

$$s(V) \leq 2d([q],[p_{\max}]) \,.$$

Sobolev gap for $SL_2(\mathbb{R})$

Theorem 4

Let $G = SL_2(\mathbb{R})$ and $V \neq \mathbb{C}$ be a unitarizable irreducible Harish-Chandra module and $[q]$ be the equivalence class of the unitary norm. Then

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- $S = S(V) = Spec_K(V) \subset \mathbb{Z}$ i.e., $V = \bigoplus_{n \in S} \mathbb{C} e_n$

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- $S = S(V) = Spec_K(V) \subset \mathbb{Z}$ i.e., $V = \bigoplus_{n \in S} \mathbb{C} e_n$
- For example, if π belongs to the unitary principal series, then $S = 2\mathbb{Z}$ or $2\mathbb{Z} + 1$ (depending on the parametrization).

Theorem 5

Let $\pi \in \widehat{G}$ be such that $\pi \neq \mathbb{1}$. Fix $m \in Spec_K(\pi)$. Then for $n \neq m$,

$$\sup_{g \in G} |\langle \pi(g) e_m^\pi, e_n^\pi \rangle| \asymp_{\pi,m} \frac{1}{\sqrt{1+|n|}},$$

except for one representation of the principal series where an additional log-factor is needed.

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except for one representation of the principal series where an additional log-factor is needed.

- Estimates of the minimal and maximal norms :

$$p_{\min}(e_n) \asymp \frac{1}{\sqrt{1+|n|}}, \quad p_{\max}(e_n) \asymp \sqrt{1+|n|}$$

- Then for any s for which $p_{\max} \lesssim p_{\min,s}$, we must have

$$p_{\max}(e_n) \lesssim p_{\min,s}(e_n) \implies (1+|n|)^{\frac{1}{2}} \lesssim (1+|n|)^{s-\frac{1}{2}}$$

- So, $s \geq 1$ which leads to $s(V) \geq 1$.

- To show $s(V) \leq 1$, we shall show that for $\epsilon > 0$,

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which leads to $d([p_{\max}], q) = \frac{1}{2}$. Hence

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- $B_{\max} = \overline{conv}(G \cdot e_m)$, for some fixed $m \in S = Spec_K(V)$. By convexity,

$$\int_G g \cdot e_m \, d\gamma(g) \in B_{\max}$$

for γ a complex Borel measure on G with total variation $\|\gamma\| \leq 1$.

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which leads to $d([p_{\max}], q) = \frac{1}{2}$. Hence

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- Enough to show that $B_{q_{\frac{1}{2}+\epsilon}} \subset B_{\max}$ i.e., $B_{\max}$ contains up to scale all sequences $\sum_{n\in S} a_n |n|^{-(\frac{1}{2}+\epsilon)} e_n$ with $\sum |a_n|^2 \leq 1$ and $\epsilon > 0$.
- $B_{\max} = \overline{conv}(G \cdot e_m)$, for some fixed $m \in S = Spec_K(V)$. By convexity,

$$\int_G g \cdot e_m \, d\gamma(g) \in B_{\max}$$

for γ a complex Borel measure on G with total variation $\|\gamma\| \leq 1$.
- Finally, choose γ such that

$$\int_G \langle g \cdot e_m, e_n \rangle d\gamma(g) \sim_{m,\pi} a_n(\pm) \; n^{-(\frac{1}{2}+\epsilon)}.$$

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$$\iota : [-1/2, 1/2] \to (\mathfrak{Norm}(V), d), \quad s \mapsto [q_s^G]$$

satisfies

$$d([q_s^G],[q_{s'}^G]) \geq |s - s'| \qquad (s, s' \in [-1/2, 1/2]) .$$

In particular, ι is injective.
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Geometry of $\mathfrak{Norm}(V)$ for $SL_2(\mathbb{R})$

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- $d([q],[p^r]) =$? Locations of the L^r-norms p^r? (Kunze-Stein Phenomenon).

Uniform finiteness of the Sobolev gap for general G

- (Finiteness questions) G- Real reductive group

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- Open questions

$$\sup_{V \in \mathrm{Irr}(\mathcal{HC})} s(V, w_V) < \infty?$$

References

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THANK YOU!