Presenting a signal in a quantum computer

Nonlinear Fourier analysis and quantum signal processing

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Matrix valued analogues ($SU(1,1)$)

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Nonlinear Fourier series ($SU(2)$)

$$G(z,n) = G(z,n-1)\frac{1}{\sqrt{1+|f_n|^2}}\begin{pmatrix} 1 & f_n z^n \\ -\overline{f_n} z^{-n} & 1 \end{pmatrix}$$

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$$\begin{pmatrix} a(z) & b(z) \\ -\overline{b}(z) & \overline{a}(z) \end{pmatrix} = \prod_{n \nearrow} \frac{1}{\sqrt{1+|f_n|^2}} \begin{pmatrix} 1 & f_n z^n \\ -\overline{f_n} z^{-n} & 1 \end{pmatrix}$$

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Multilinear expansion

$$S\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \prod_{n\nearrow}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & f_n z^n \\ -\overline{f_n} z^{-n} & 0 \end{pmatrix}\right]$$

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$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_n \begin{pmatrix} 0 & f_n z^n \\ -\overline{f_n} z^{-n} & 0 \end{pmatrix}$$

$$+ \sum_{n_1<n_2} \begin{pmatrix} -f_{n_1}\overline{f_{n_2}} z^{n_1-n_2} & 0 \\ 0 & -\overline{f_{n_1}} f_{n_2} z^{-n_1+n_2} \end{pmatrix}$$

$$+ \sum_{n_1<n_2<n_3} \begin{pmatrix} 0 & -f_{n_1}\overline{f_{n_2}} f_{n_3} z^{n_1-n_2+n_3} \\ -\overline{f_{n_1}} f_{n_2}\overline{f_{n_3}} z^{-n_1+n_2-n_3} & 0 \end{pmatrix} + \sum_{n_1<n_2<n_3<n_4} \cdots$$

Plancherel for outer a

$$\left[\prod_n (1 + |f_n|^2)\right]^{\frac{1}{2}} a(z) = 1 + a_{-2} z^{-2} + a_4 z^{-4} \dots$$

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$$\sum_n |f_n|^2 \sim \int_{\mathbb{T}} |b(z)|^2$$

Layer stripping

$$\begin{pmatrix} a_{\geq 0}(z) & b_{\geq 0}(z) \\ -b_{\geq 0}{}^{*}(z) & a_{\geq 0}{}^{*}(z) \end{pmatrix} = \prod_{0 \leq n \nearrow} \frac{1}{\sqrt{1+|f_n|^2}} \begin{pmatrix} 1 & f_n z^n \\ -\overline{f_n} z^{-n} & 1 \end{pmatrix}$$

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$$\frac{1}{\sqrt{1+|f_0|^2}} \begin{pmatrix} 1 & -f_0 \\ \overline{f_0} & 1 \end{pmatrix} \begin{pmatrix} * & b_{\geq 0}(0) \\ * & a_{\geq 0}{}^*(0) \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

Layer stripping

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$$f_0 = \frac{b_{\geq 0}(0)}{a^*_{\geq 0}(0)}$$

Riemann-Hilbert

$$\begin{pmatrix} a(z) & b(z) \\ -b^*(z) & a^*(z) \end{pmatrix} = \begin{pmatrix} a_{<0}(z) & b_{<0}(z) \\ -b_{<0}{}^*(z) & a_{<0}{}^*(z) \end{pmatrix} \begin{pmatrix} a_{\geq 0}(z) & b_{\geq 0}(z) \\ -b_{\geq 0}{}^*(z) & a_{\geq 0}{}^*(z) \end{pmatrix}$$

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$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a_+^* & -b_+ \\ b_+^* & a_+ \end{pmatrix} = \begin{pmatrix} a_- & b_- \\ -b_-{}^* & a_-{}^* \end{pmatrix}$$

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$$a_+^* + P_{\mathbb{D}} \left(\frac{b}{a} b_+^* \right) = \frac{a_-(\infty)}{a(\infty)} = \frac{1}{a_+(\infty)}$$

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$$-\frac{b^*}{a^*} a_+^* + b_+^* = -\frac{b_-^*}{a^*}$$

$$-P_{\mathbb{D}^*}\left(\frac{b^*}{a^*} a_+^*\right) + b_+^* = 0$$

More more Riemann-Hilbert

$$A = a_+^* a_+(\infty), \;\; B = b_+ a_+(\infty)$$

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$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & P_{\mathbb{D}} \circ \frac{b}{a} \\ -P_{\mathbb{D}^*} \circ \frac{b^*}{a^*} & 0 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

A new look at Fourier series

$$a_-nz^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_{n-1}z^{n-1} + a_nz^n$$

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$$a_-nz^{-n} + a_{-n+1}z^{-n+1} + \cdots + a_{n-1}z^{n-1} + a_nz^n$$

$$= ((\ldots((a_-nz^{-1} + a_{-n+1})z^{-1} + \cdots + a_{n-1})z^{-1} + a_n)z^n$$

A new look at nonlinear Fourier series

$$\begin{pmatrix} a(z) & b(z) \\ -\overline{b}(z) & \overline{a}(z) \end{pmatrix} = \prod_{-N \le n \le N \nearrow} \frac{1}{\sqrt{1+|f_n|^2}} \begin{pmatrix} 1 & f_n z^n \\ -\overline{f_n} z^{-n} & 1 \end{pmatrix}$$

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$$\begin{pmatrix} a(z)z^n & b(z) \\ -\overline{b}(z) & \overline{a}(z)z^{-n} \end{pmatrix} =$$

$$\prod_{-N\le n\le N \nearrow} \left[\frac{1}{\sqrt{1+|f_n|^2}} \begin{pmatrix} 1 & f_n \\ -\overline{f_n} & 1 \end{pmatrix} \begin{pmatrix} z^{\frac{1}{2}} & 0 \\ 0 & z^{-\frac{1}{2}} \end{pmatrix} \right]$$

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$$Z^{-n} Z(\ldots(Z(ZFZ^{-1}F)Z^{-1})F\ldots)Z^{-1}FZ^n \tag{1}$$

Quantum signal processing

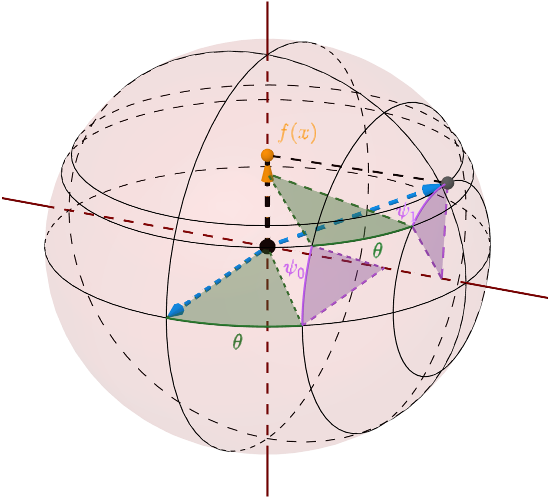

Figure: Illustration of QSP

References

Joint work with Michel Alexis, Lin Lin, Gevorg Mnatsakanyan, Jiasu Wang

Thank you.