Exceptional directions in hyperbolic FPP

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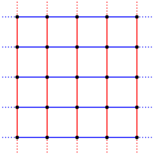
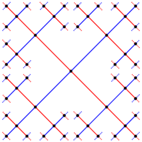
Background geometry

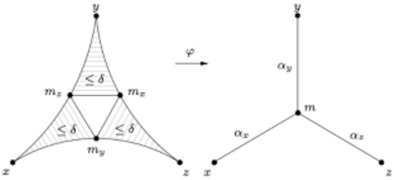
Cayley graph: (G, S) group with finite generating set.
$\mathcal{V}(\Gamma) = \{g | g \in G\}$, $\mathcal{E}(\Gamma) = \{(g, h) | g^{-1}h \in S\}$.

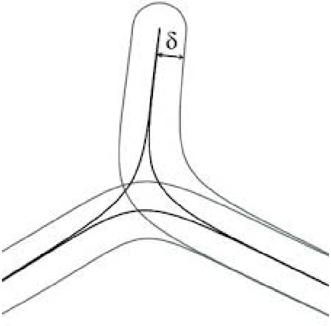




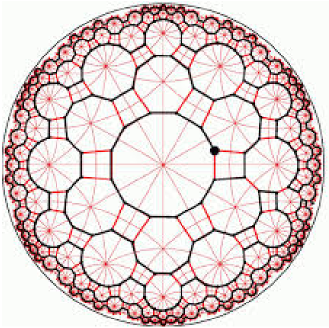
Hyperbolic groups

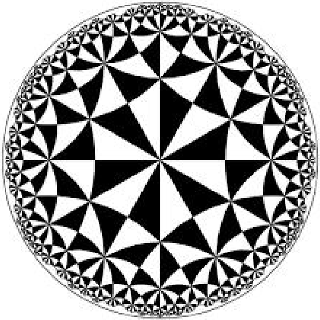
A geodesic metric space (X, d) is $\delta-$hyperbolic if for all
$x, y, z \in X$,
$[x, y] \subset N_\delta([x, z] \cup [y, z])$.

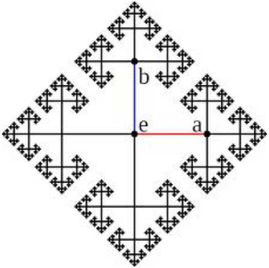




G is hyperbolic if $\Gamma(G, S)$ is hyperbolic with respect to some (any) finite S.
Such groups are ubiquitous.



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FPP

$\Gamma = (\mathcal{V}, \mathcal{E})$ a graph.
$\rho =$ probability distribution on $\mathbb{R}_+ := (0, \infty)$
$(\Omega, \mathbb{P}) = (\mathbb{R}_+, \rho)^E$.
Random variables $X_e : \Omega \to (0, \infty)$ given by $X_e(\omega)$ iid with law ρ.
$\gamma = \{e_1, \cdots e_k\}$–edge path. For $\omega \in (\Omega, \mathbb{P})$, $\ell_\omega(\gamma) := \sum_{e \in \gamma} \omega(e)$.
$d_\omega(x, y) := \inf_\gamma \ell_\omega(\gamma)$.
$\Upsilon(x, y)(\omega)$ – ω-geodesic between x and y–**FPP-geodesic** between x and y.
$T(x, y)(\omega) = d_\omega(x, y)$ –**first passage time** between x and y.
Widely studied for $\mathbb{Z}^n$.
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Parametrizing directions

Key feature of hyperbolic group G = boundary ∂G.
Use ∂G to parametrize directions.
Measure ν on ∂G = weak limit of uniform measure on balls–Patterson-Sullivan measure.
Almost every with respect to direction $\Rightarrow$ Patterson-Sullivan measure.
Almost every $\omega-$geodesic $\Rightarrow$ $(\Omega, \mathbb{P})$.

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Older Results: 1) Direction Exists

Say that a path σ starting at o has direction ξ if the only accumulation point of σ in ∂G is ξ.

Theorem ((1) Direction Exists)

Given $\xi \in \partial G$, $x_n \to \xi$ for a.e. $\omega \in (\Omega, \mathbb{P})$, the sequence of $\omega-$geodesics $[o, x_n]_\omega$ from o to x_n converges (up to subsequence) to a ray $[o, \xi)_\omega$ having direction ξ.

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2) Velocity Exists and is constant

Expected passage time from x to y is given by $\mathbb{E}T(x,y) := \int_\Omega T(x,y)(\omega)\, d\mathbb{P}$.
For $\xi \in \partial G$, define the velocity in the direction of ξ to be

$$v(\xi) := \lim_{n\to\infty} \frac{\mathbb{E}T(o,x_n)}{d(o,x_n)},$$

provided the limit exists.
Assumption: ρ has no atom at 0 and $\exists a > 0$ such that $\int e^{ax^2}\, d\rho(x) < \infty$.

Theorem ((2) Velocity Exists)

For a.e. $\xi \in \partial G$, the velocity $v(\xi)$ in the direction of ξ exists. Further, $v(\xi)$ is constant almost everywhere.

Not possible to upgrade a.e. $\xi \in \partial G$ to every $\xi \in \partial G$.

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(3) Linear growth of variance

Second order behavior of passage time $T(1, x_n)$ along geodesics grows linearly.

Theorem

There exists $0 < C_1 < C_2 < \infty$ *such that*

$$C_1 n \leq \mathrm{Var}(T(1, x_n)) \leq C_2 n.$$

Resolves conjecture of Benjamini, Tessera and Zeitouni affirmatively.

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4) Coalescence

Two semi-infinite paths σ_1, σ_2 are said to **coalesce** if beyond some x_0, they coincide.

Theorem ((4) Coalescence of ω−geodesics)

Given direction ξ, for a.e. $\omega \in (\Omega, \mathbb{P})$, $o_1, o_2 \in G$ ω−geodesic rays $[o_1, \xi)_\omega$ and $[o_2, \xi)_\omega$ a.s. coalesce.

Random backward tree $T(\xi, \omega)$: union of all random geodesics directed towards $\xi \in \partial G$.
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Given direction ξ, for a.e. $\omega \in (\Omega, \mathbb{P})$, $o_1, o_2 \in G$ ω−geodesic rays $[o_1, \xi)_\omega$ and $[o_2, \xi)_\omega$ a.s. coalesce.

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Complete Backward Trees

Definition

A backward tree $T(\xi, \omega)$ has **complete ends** if for all $\xi' \neq \xi$, there exists $p \in [1, \xi)_\omega$ such that $(\xi', p]_\omega \subset T(\xi, \omega)$.

Assumption: edge weights are sub-exponential. $T(\xi, \omega)$ contains a union of bi-infinite geodesics $\{(\xi', \xi)_\omega\}$, where ξ' ranges over $\partial G \setminus \{\xi\}$.

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Exceptional directions

Question

Are there points $\xi \in \partial G$, such that there are disjoint $\omega-$geodesics asymptotic to ξ?

Such points ξ are called **exceptional directions.** (depends on ω).

Question

How many disjoint random geodesics are asymptotic to ξ?

Called **multiplicity** of ξ.

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Definition

Let (X, d_X) and (Y, d_Y) be hyperbolic metric spaces and let $i : Y \to X$ denote a proper embedding. Let ∂X, ∂Y be their (Gromov) boundaries. Also, let $\widehat{X}$ and $\widehat{Y}$ denote their Gromov compactifications. We say that the triple (X, Y, i) admits a **Cannon-Thurston map**, if i extends continuously to $\hat{i} : \widehat{Y} \to \widehat{X}$.

$(T(\omega), d^s_\omega)$-back/forward tree with intrinsic simplicial metric d^s_ω.

Theorem

$(\Gamma, T(\omega), i_\omega)$ admits a surjective Cannon-Thurston map for a full measure subset.

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$(\Gamma, T(\omega), i_\omega)$ *admits a surjective Cannon-Thurston map for a full measure subset.*

Existence and measure of exceptional directions

Theorem

Let G be a hyperbolic group such that G is not virtually free. Then for a.e. $\omega \in \Omega$, exceptional directions in ∂G exist and are dense in it.

Theorem

For $\mathbb{P}$-a.e. ω, exceptional directions have zero Patterson-Sullivan measure. In fact, they have Hausdorff dimension strictly less than that of ∂G.

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Cardinality of exceptional directions

Theorem

Let G be a hyperbolic group such that $\dim_t \partial G > 1$. Then for a full measure subset and for $T(\omega)$ either the forward or backward tree, the number of exceptional directions must be uncountable.

Theorem

Let G be a hyperbolic group acting cocompactly on the hyperbolic plane $\mathbf{H}^2$ such that the Cayley graph Γ is embedded in $\mathbf{H}^2$. Then for a full measure subset, and for both forward and backward trees $T(\omega)$, the number of exceptional directions must be countable.

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Theorem

Let G be hyperbolic such that $\dim_t \partial G = n - 1$. Let $T(\omega)$ be a random backward or forward tree. Then for any ω in a full measure set there exists an exceptional direction $z \in \partial G$ with multiplicity at least n.

Theorem

There exists $k_0 = k_0(\Gamma, \delta)$ such for a full measure set there does not exist any direction $\xi \in \partial G$ with multiplicity more than k_0, i.e. there do not exist more than k_0 distinct ω-geodesics from 1 in the direction ξ.

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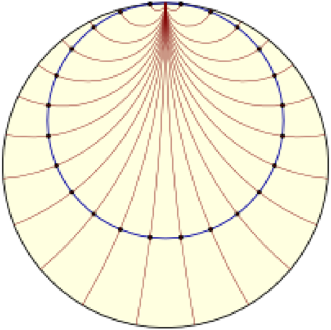
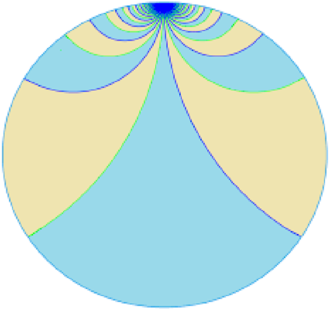
From trees inside to random partitions on boundary





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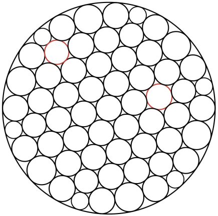

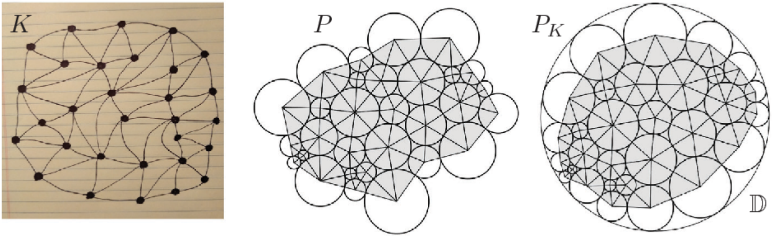

THANK YOU!