UNIVERSITÄT PADERBORN

Quantum Resonances and Scattering Poles

Joachim Hilgert

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Outline

What is a quantum resonance?

resonance: pole of meromorphically continued resolvent $z \mapsto (D - z)^{-1}$

quantum: D related to a quantum Hamiltonian

Example (Free particle)

Laplace-Beltrami operator Δ on a complete Riemannian manifold X.

$$\Delta : C^\infty(X) \to C^\infty(X) \text{ essentially self-adjoint}$$
$$\Delta : L^2(X, d_{\mathrm{vol}}) \to L^2(X, d_{\mathrm{vol}}) \text{ s.a. positive}$$

$$(\Delta - z)^{-1} : C_c^\infty(X) \to \mathcal{D}'(X) \quad \leftrightsquigarrow \quad \text{Schwartz kernel}$$

Meromorphically continue the family of Schwartz kernels from $\mathbb{C} \setminus \mathrm{spec}(\Delta) \supseteq \mathbb{C} \setminus [0, \infty[$ and determine the poles!

Riemannian symmetric spaces of non-compact type

Problem ($X = G/K$ Riemannian symmetric spaces of nc type)

a) *Show that the (modified) resolvent $(\Delta - \rho^2 - \zeta^2)^{-1}$ has meromorphic continuation to $\mathbb{C}$.*

b) *Determine poles and residues.*

solved for

rank 1 [Miatello-Will '00], [H-Pasquale '09]

most rank 2 cases [H-Pasquale-Przebinda '17]

What is a scattering pole?

scattering: comparing asymptotics of evolutions for different time directions

scattering operator: operator intertwining asymptotic solutions of a (wave) equation.

scattering pole: parameterized (e.g. by frequencies) family of equations

$\rightsquigarrow$ parameterized family of scattering operators

if meromorphic

$\rightsquigarrow$ a pole of that family (after normalization)

Symmetric spaces of non-compact type

Example ($X = G/K$ Riemannian symmetric space of nc type)

Wave equation

- $\rightsquigarrow$ Laplace eigenfunctions
- $\rightsquigarrow$ further splitting of spectral lines
- $\rightsquigarrow$ joint $\mathbb{D}(X)$-eigenfunctions (inv. diff. ops)
- $\rightsquigarrow$ spectral parameters $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$
  (from $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ max. abelian)
- $\rightsquigarrow$ function spaces $\mathcal{H}_\lambda$ on the maximal boundary G/P
  (spherical principal series representations)
- $\rightsquigarrow$ standard (Knapp-Stein) intertwining operators
  meromorphic family on $\mathfrak{a}^*_{\mathbb{C}}$ (see e.g. [Wallach '92])

Physics mantra: resonances $\equiv$ scattering poles

Explanation in [Borthwick '16]

Resonances: $(\partial_t^2 + \Delta - \frac{1}{4})u = e^{i\xi t}\phi$

$$R\Big(\frac{1}{2}+i\xi\Big) = \Big(\Delta-\Big(\frac{1}{2}+i\xi\Big)\Big(\frac{1}{2}-i\xi\Big)\Big)^{-1} = \Big(\Delta-\frac{1}{4}-\xi^2\Big)^{-1}$$

$$\Big(\Delta - \frac{1}{4}\Big)u = \Big(\Delta - \frac{1}{4} - \xi^2\Big)u + \xi^2 u = e^{i\xi t}\phi + \xi^2 u$$

$$u = e^{i\xi t}R\Big(\tfrac{1}{2} + i\xi\Big)\phi$$

Scattering poles: $(\Delta - s(1-2))P(s)\psi = 0$ Poisson trafo

$$2s-1)P(s)\psi \sim c^{1-s}\psi + c^s\phi_s$$

$$S(s) : \psi \mapsto \phi_s$$

Older results

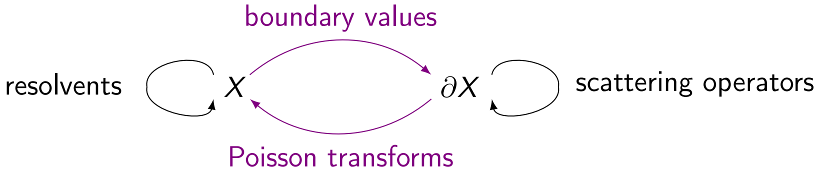
Problem: give a mathematically rigorous explanation

Results:
- hyperbolic surfaces [Guillopé-Zworski '97]
- asymptotically (real) hyperbolic manifolds [Borthwick-Perry '02], [Guillarmou '05]
- rank 1 Riemannian symmetric spaces [Hansen-H-Parthasarathy '19]

Quantum resonances are roughly one half of the scattering poles. The other half are poles of the unnormalized scattering operators.

Technical tools

Harmonic analysis: Plays a role in all cases treated so far



Melrose type microlocal analysis: This has been used in the case of asymptotically hyperbolic manifolds

Bunke-Olbrich theory: We use this for the case of rank one convex cocompact locally symmetric spaces (see below)

Bunke-Olbrich theory for classical rank 1 spaces

Γ: torsion free discrete convex cocompact subgroup of G

$\Gamma\backslash X$: locally symmetric space, manifold with boundary $\Gamma\backslash\Omega$

Ω: complement of the limit set Λ of Γ in ∂X

[Bunke-Olbrich '12] contains a comparison between the spectral theories of Δ_X and $\Delta_{\Gamma\backslash X}$ – including resolvents and scattering operators.

Key tool: *pullback* of distributions on $\Gamma\backslash\Omega$ to Γ-invariant distributions on Ω and *extension* to Γ-invariant distributions on ∂X.

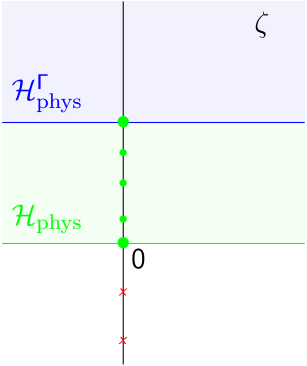
Consequences of the Bunke-Olbrich theory

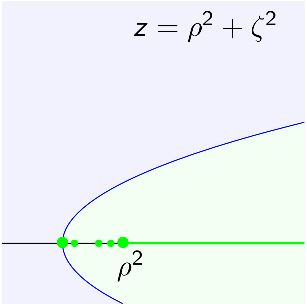
Combining the Bunke-Olbrich comparison for resolvent and scattering kernels with the precise results for symmetric spaces from [HHP19] yields:

Theorem (H-Delarue '24)

Away from an exceptional set of spectral parameters (related to singular Poisson transforms) quantum resonances are essentially one half of the scattering poles.

- The results for asymptotically hyperbolic spaces also contain precise statements about multiplicities.
- It seems plausible that the techniques using Gohberg-Sigal normal forms employed by Guillarmou to determine multiplicities can be used also in our case.





The scattering poles *not* in the physical halfplane are the resonances

An abstract lemma on elliptic operators

A problem we face is to deal with the singularities of the Schwartz kernels along the diagonal. It is solved using

Lemma (H-Delarue '24)

Let M be a smooth Riemannian manifold and P an elliptic operotor of order > 0 on M. Suppose the resolvent $(P - z)^{-1} : L^2(M) \to L^2(M)$ is defined on an open set $U \subseteq \mathbb{C}$ and extends to an open connected set $V \supseteq U$ as a meromorphic family of operators $R(z) : C_c^\infty(M) \to \mathcal{D}'(M)$. Then $z_0 \in V$ is a pole of the Schwartz kernel $r(z) \in \mathcal{D}'(M \times M)$ iff it is a pole of $r(z)|_{M\times M\setminus \mathrm{diag}_M} \in \mathcal{D}'(M \times M \setminus \mathrm{diag}_M)$.

Higher rank symmetric spaces

Scattering operators: Knapp-Stein operators – see [STS76] for the scattering interpretation

Invariant differential operators $D \in \mathbb{D}(X)$: Replace the Laplacian

Harish-Chandra isomorphism $\Gamma : \mathbb{D}(X) \to \mathbb{C}[\mathfrak{a}_\mathbb{C}]^W$: $\Gamma(D)(\zeta)$ replaces the quadratic polynomial $\rho^2 + \zeta^2$

Resolvent: ??? – what is the resolvent of a finitely generated commutative algebra of (differential) operators? But see [GGHW24] for the solution of a related problem!

Higher rank locally symmetric spaces

Missing ingredients:

Bunke-Olbrich theory for Anosov groups replacing convex cocompactness

Scattering theory in the spirit of Semenov-Tjan-Šanskiĭ using the multi-temporal wave equation

Thank you!

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