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and Research Bhopal, India<br>
th R. P. Sarkar)<br>
Sroup (APRG) Seminar<br>
nematics<br>
Science<br>
025 A theorem of Strichartz for multipliers on homogeneous trees

Sumit Kumar Rano

### Indian Institute of Science Education and Research Bhopal, India

(Based on a joint work with R. P. Sarkar)

Analysis and Probability Research Group (APRG) Seminar

Department of Mathematics

Indian Institute of Science

29th January, 2025

[A Theorem of Roe and Strichartz on](#page-2-0)  $\mathbb{R}^n$ 

[Homogeneous Trees](#page-14-0)

omogeneous Trees<br>ogeneous Trees<br>2 [Strichartz's Theorem for the Laplacian on Homogeneous Trees](#page-28-0)

[Strichartz's Theorem for Multipliers on Homogeneous Trees](#page-62-0)

[Notable Consequences](#page-104-0)

# <span id="page-2-0"></span>Strichartz on  $\mathbb{R}^n$ [A Theorem of Roe and Strichartz on](#page-2-0)  $\mathbb{R}^n$

# A brief history

**■ J. Roe, 1980** : Let  $\{f_k\}_{k \in \mathbb{Z}}$  be a doubly infinite sequence of functions on R such that

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\frac{d}{dx}f_k = f_{k+1} \text{ and } \|f_k\|_{L^{\infty}(\mathbb{R})} \leq M, \text{ for all } k \in \mathbb{Z}.
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**■** If  $\Delta_{\mathbb{R}^n} f_k = A f_{k+1}$  for some  $A \in \mathbb{C}^\times$ , then  $\Delta_{\mathbb{R}^n} f_0 = -|A| f_0$ .

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### **Question**

Does a precise analogue of Strichartz's theorem apply to the combinatorial Laplacian  $\mathscr L$  on a homogeneous tree  $\mathscr X$  ?

# <span id="page-14-0"></span>[Homogeneous Trees](#page-14-0)

I

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- **The boundary**  $\Omega$  **is identified with the set of all infinite geodesic rays** starting at o.

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\mathscr{L} u(x) = u(x) - \frac{1}{q+1} \sum_{y:d(x,y)=1} u(y).
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# <span id="page-28-0"></span> $\Gamma \nsubseteq \text{On } \mathcal{X}$ [Strichartz's Theorem for](#page-28-0)  $\mathscr L$  on  $\mathscr X$

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\phi_z(x) = \int_{\Omega} \rho^{1/2+iz}(x,\omega) d\nu(\omega),
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**\*** Fact :  $\phi_z \in L^\infty(\mathcal{X})$  if and only if  $z \in \mathbb{C}$  satisfies  $|\Im z| \leq 1/2$ .
## L<sup>∞</sup>-point spectrum of  $\mathscr L$

- $\bullet$  L<sup>∞</sup>-point spectrum of  $\mathscr{L}$  : { $\gamma(z)$  :  $z \in \mathbb{C}$  and  $|\Im z| \leq 1/2$ }.
- $\bullet$  Unlike the L<sup>∞</sup>-point spectrum of  $\Delta_{\mathbb{R}^n}$  which is the one-dimensional interval  $(-\infty, 0]$ , the  $L^{\infty}$ -point spectrum of  ${\mathscr L}$  is an elliptic region in the complex plane centered around the point 1.



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 $\bullet$  Choose two points  $z_1$ ,  $z_2$  in { $z \in \mathbb{C}$  :  $|\Im z| \leq 1/2$ } such that

 $\gamma(z_1) \neq \gamma(z_2)$  and  $|\gamma(z_1)| = |\gamma(z_2)| = 1$ .



- $|z| \le 1/2$ } such that<br> $|z_1| = |\gamma(z_2)| = 1.$ <br> $f_k\}_{k \in \mathbb{Z}}$  as follows :<br> $|z_2|^k \phi_{z_2}(x), \quad x \in \mathcal{X}.$  $\bullet$  Choose two points  $z_1$ ,  $z_2$  in  $\{z \in \mathbb{C} : |\Im z| \leq 1/2\}$  such that  $\gamma(z_1) \neq \gamma(z_2)$  and  $|\gamma(z_1)| = |\gamma(z_2)| = 1$ .
- $\bullet$  Consider the doubly infinite sequence  $\{f_k\}_{k\in\mathbb{Z}}$  as follows :
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f_k(x)=\gamma(z_1)^k\phi_{z_1}(x)+\gamma(z_2)^k\phi_{z_2}(x),\quad x\in\mathcal{X}.
$$

\* 
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||f_k||_{L^{\infty}(\mathcal{X})} \le ||\phi_{z_1}||_{L^{\infty}(\mathcal{X})} + ||\phi_{z_2}||_{L^{\infty}(\mathcal{X})} \le 2.
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f_k(x)=\gamma(z_1)^k\phi_{z_1}(x)+\gamma(z_2)^k\phi_{z_2}(x),\quad x\in\mathcal{X}.
$$

\* 
$$
||f_k||_{L^{\infty}(\mathcal{X})} \le ||\phi_{z_1}||_{L^{\infty}(\mathcal{X})} + ||\phi_{z_2}||_{L^{\infty}(\mathcal{X})} \le 2.
$$

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\mathscr{L}f_k(x) = \gamma(z_1)^k \mathscr{L} \phi_{z_1}(x) + \gamma(z_2)^k \mathscr{L} \phi_{z_2}(x) = f_{k+1}(x).
$$

- $|z| \le 1/2$ } such that<br>  $|z_1| = |\gamma(z_2)| = 1.$ <br>  $f_k]_{k \in \mathbb{Z}}$  as follows :<br>  $|z_2|^k \phi_{z_2}(x), \quad x \in \mathcal{X}.$ <br>  $\le 2.$ <br>  $\phi_{z_2}(x) = f_{k+1}(x).$ <br>  $\phi_{z_1}(x) = f_{k+1}(x).$ <br>  $\phi_{z_2}(x) = f_{k+1}(x).$  $\bullet$  Choose two points  $z_1$ ,  $z_2$  in  $\{z \in \mathbb{C} : |\Im z| \leq 1/2\}$  such that  $\gamma(z_1) \neq \gamma(z_2)$  and  $|\gamma(z_1)| = |\gamma(z_2)| = 1$ .
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**<sup>★</sup>** Therefore  $\{f_k\}_{k\in\mathbb{Z}}$  satisfies all the hypothesis of Strichartz's theorem.

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- **<sup>★</sup>** Therefore  $\{f_k\}_{k\in\mathbb{Z}}$  satisfies all the hypothesis of Strichartz's theorem.
- $\cdot$  However,  $f_0$  fails to be an eigenfunction of  $\mathscr{L}$ .
- $\bullet$  Notations for today : Let  $1 < p \le 2$ . Then
	- $p'$  denotes the conjugate exponent  $p/(p-1)$ .

\* 
$$
\delta_{p'} = \frac{1}{p'} - \frac{1}{2}
$$
.

$$
\bullet \ \ S_p = \{z \in \mathbb{C} : |\Im z| \leq |\delta_{p'}| \}.
$$



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**Assumption** :  $p' = \infty$  when  $p = 1$ .

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\bullet \ \delta_{\infty}=-1/2 \text{ and } S_1=\{z\in \mathbb{C}: |\Im z|\leq 1/2\}.
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- **Assumption** :  $p' = \infty$  when  $p = 1$ .
- $\bullet$   $\delta_{\infty} = -1/2$  and  $S_1 = \{z \in \mathbb{C} : |\Im z| < 1/2\}.$
- **In Observation :**  $\delta_2 = 0$  and  $S_2 = \mathbb{R}$ .
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	- +  $\phi_z \in L^{2,\infty}(\mathcal{X})$  if and only if  $z \in \mathbb{R} \setminus (\tau/2)\mathbb{Z}$ .











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= A  $f_{k+1}$ , for all  $k \in \mathbb{Z}$ ,<br>  $\infty$ ). Then  $\mathscr{L}f_0 = \gamma(\tau/2 + i\delta_{\infty})f_0$ . **E** S. K. Rano, 2022 : Let  $\{f_k\}_{k \in \mathbb{Z}}$  be a doubly infinite sequence of functions on  $\mathscr X$  satisfying

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	- $I$  If  $\mathscr{L}f_{-k} = A f_{-k+1}$  for all  $k \in \mathbb{N}$ , where  $A \in \mathbb{C}$  satisfies  $|A| = \gamma(\tau/2 + i\delta_{p'})$ , then  $\mathscr{L} f_0 = \gamma(\tau/2 + i\delta_{p'}) f_0$ .

-infinite sequence of functions on<br>  $= A f_{k+1}$ , for all  $k \in \mathbb{Z}$ ,<br>  $b = \frac{2\sqrt{q}}{q+1}$ , **E** S. K. Rano, 2022 : Let  $\{f_k\}_{k\in\mathbb{Z}}$  be a bi-infinite sequence of functions on  $\mathscr X$  satisfying

$$
||f_k||_{L^{2,\infty}(\mathcal{X})} \leq M \text{ and } \mathscr{L}f_k = A f_{k+1}, \text{ for all } k \in \mathbb{Z},
$$

where  $A \in \mathbb{C}$  is such that

$$
|A| \in (1 - b, 1 + b), \quad b = \frac{2\sqrt{q}}{q+1},
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### **Question**

-infinite sequence of functions on<br>  $= A f_{k+1}$ , for all  $k \in \mathbb{Z}$ ,<br>  $\bigg)$ ,  $b = \frac{2\sqrt{q}}{q+1}$ ,<br>
mials of  $\mathcal{L}$ , the spherical averages What happens if we replace  $\mathscr L$  with polynomials of  $\mathscr L$ , the spherical averages on  $\mathscr X$ , or the heat operator on  $\mathscr X$  ?

**E** S. K. Rano, 2022 : Let  $\{f_k\}_{k\in\mathbb{Z}}$  be a bi-infinite sequence of functions on  $\mathscr X$  satisfying

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### **Question**

What happens if we replace  $\mathscr L$  with polynomials of  $\mathscr L$ , the spherical averages on  $\mathcal X$ , or the heat operator on  $\mathcal X$  ?

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mials of  $\mathcal{L}$ , the spherical averages<br>
the above results for multipliers<br>  $\big)$ ■ We shall specifically focus on extending the above results for multipliers when  $1 < p < 2$ .

## <span id="page-62-0"></span>[Strichartz's Theorem for Multipliers](#page-62-0)



## Fourier transforms on  $\mathcal X$

upported radial function  $f$  on  $\mathcal{X}$  is<br>  $\kappa$ ), where  $z \in \mathbb{C}$ .  $\bullet$  The spherical transform  $\widehat{f}$  of a finitely supported radial function f on  $\mathcal X$  is defined by the formula

$$
\widehat{f}(z) = \sum_{x \in \mathcal{X}} f(x) \phi_z(x), \text{ where } z \in \mathbb{C}.
$$

## Fourier transforms on  $X$

upported radial function  $f$  on  $\mathcal{X}$  is<br>  $(x)$ , where  $z \in \mathbb{C}$ .<br>  $(z) = \hat{f}(z + \tau)$ . **The spherical transform**  $\widehat{f}$  of a finitely supported radial function f on  $\mathcal X$  is defined by the formula

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**Example 1** Symmetry and Periodicity :  $\hat{f}(z) = \hat{f}(-z) = \hat{f}(z + \tau)$ .

### Fourier transforms on  $\mathcal X$

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ormula<br>  $p^{1/2 + iz}(x, \omega)$ . **The Helgason-Fourier transform**  $\tilde{f}$  of a finitely supported function f on  $\mathcal{X}$ is a function on  $\mathbb{C} \times \Omega$  defined by the formula

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\widetilde{f}(z,\omega)=\sum_{x\in\mathcal{X}}f(x)\,\,p^{1/2+iz}(x,\omega).
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II  $ω ∈ Ω$ .  $\bullet$  The Helgason-Fourier transform  $\widetilde{f}$  of a finitely supported function f on  $\mathcal X$ is a function on  $\mathbb{C} \times \Omega$  defined by the formula

$$
\widetilde{f}(z,\omega)=\sum_{x\in\mathcal{X}}f(x)\,\,p^{1/2+iz}(x,\omega).
$$

**Periodicity** :  $\tilde{f}(z, \omega) = \tilde{f}(z + \tau, \omega)$ .

If f is radial, then  $\tilde{f}(z, \omega) = \hat{f}(z)$ , for all  $\omega \in \Omega$ .

**Example 3 Schwartz spaces**  $\mathcal{S}_p(\mathcal{X})$  : Space of all functions  $\phi$  on  $\mathcal{X}$  for which

$$
\nu_{p,m}(\phi)=\sup_{x\in\mathcal{X}}\left(1+|x|\right)^{m}q^{|x|/p}|\phi(x)|<\infty,\quad\text{for all }m\in\mathbb{Z}_{+}.
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**Example 3 Schwartz spaces**  $\mathcal{S}_p(\mathcal{X})$  : Space of all functions  $\phi$  on  $\mathcal{X}$  for which

spaces on 
$$
\mathcal{X}, 1 \leq p \leq 2
$$

\nartz spaces  $\mathcal{S}_p(\mathcal{X})$ : Space of all functions  $\phi$  on  $\mathcal{X}$  for which  $\nu_{p,m}(\phi) = \sup_{x \in \mathcal{X}} (1 + |x|)^m q^{|x|/p} |\phi(x)| < \infty$ , for all  $m \in \mathbb{Z}_+$ .

\n) forms a Fréchet space w.r.t. the countable semi-norms  $\nu_{p,m}(\cdot)$ .

 $\bullet$   $\mathcal{S}_p(\mathcal{X})$  forms a Fréchet space w.r.t. the countable semi-norms  $\nu_{p,m}(\cdot)$ .

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$$

 $\bullet S_p(\mathcal{X})$  forms a Fréchet space w.r.t. the countable semi-norms  $\nu_{p,m}(\cdot)$ .

#### Definition

Let m be an even,  $\tau$ -periodic, bounded measurable function on  $\mathbb{R}$ . An operator Θ defined as

$$
\Theta f(x) = c_x \int\limits_{\mathbb{T}} \int\limits_{\Omega} m(z) \ \widetilde{f}(z,\omega) \ p^{1/2 - iz}(x,\omega) \ |c(z)|^{-2} \ d\nu(\omega) \ dz,
$$

unctions  $\phi$  on  $\mathscr X$  for which<br>  $\phi(x)| < \infty$ , for all  $m \in \mathbb{Z}_+$ .<br>
countable semi-norms  $\nu_{p,m}(\cdot)$ .<br>
surable function on  $\mathbb R$ . An<br>  $\pi^i z(x,\omega) |c(z)|^{-2} d\nu(\omega) dz$ ,<br>
mbol m(z) if, for every semi-norm<br>  $\nu_{p,m_1}(\cdot)$  of  $\mathscr$ is said to be a multiplier on  $S_p(\mathcal{X})$  with symbol m(z) if, for every semi-norm  $\nu_{\rho,m_2}(\cdot)$  of  $\mathscr{S}_{\rho}(\mathscr{X})$ , there exists a semi-norm  $\nu_{\rho,m_1}(\cdot)$  of  $\mathscr{S}_{\rho}(\mathscr{X})$  and a constant  $C_{m_1,m_2} > 0$  such that

$$
\nu_{p,m_2}(\Theta f) \leq C_{m_1,m_2} \nu_{p,m_1}(f), \quad \text{for all } f \in \mathcal{S}_p(\mathcal{X}).
$$

**The space**  $\mathcal{H}(S_p)$  : Space of all such functions  $\psi : S_p \to \mathbb{C}$  which satisfy the following properties:


# Characterization of multipliers on the Schwartz spaces

- **The space**  $\mathcal{H}(S_p)$  : Space of all such functions  $\psi : S_p \to \mathbb{C}$  which satisfy the following properties:
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## Characterization of multipliers on the Schwartz spaces

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Proposition ( S. K. Rano and R. P. Sarkar ; Math. Z. , 2025 )

Let  $1 \le p < 2$ . Then the following are equivalent.

(a) The operator  $\Theta$  is a multiplier on  $\mathcal{S}_p(\mathcal{X})$  with symbol m(z).

(b) m is in  $\mathcal{H}(S_p)$ .

**The Laplacian**  $\mathscr L$  is a multiplier on  $\mathscr S_p(\mathscr X)$  with symbol  $\gamma(z)$ .



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\psi_z(n)=\frac{1}{\#B(o,n)}\sum_{j=0}^n\#S(o,j)\ \phi_z(j),\quad\text{for all}\ \ n\in\mathbb{Z}_+.
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## Strichartz's theorem on  $\mathscr L$  revisited

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 $\blacktriangleright$  Difficulty : The range of m may intersect  $\{w \in \mathbb{C} : |w| = |A|\}$  at more than one point.



 $\bullet$  Multiplier :  $I - \mathcal{L}$ . Symbol :  $m(z) = 1 - \gamma(z)$ .



Math. Z., 2025)<br>
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Math. Z., 2025)<br>
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- E Key examples : Polynomials of  $\mathscr{L}$ , the spherical and the ball averages on  $\mathscr{X}$ , the heat operator on  $\mathscr{X}$ .

**Math. Z., 2025** )<br>
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where  $-\tau/2 < \alpha_m \leq \tau/2$  are distinct and  $|\Psi \circ \gamma(\alpha_m + i\delta_{p'})| = |A|.$ 

Math. Z., 2025)<br>
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- (a)  $|A| = \min\{|\Psi \circ \gamma(z)| : z \in S_p\}.$
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Then  $f_0$  can be uniquely written as

$$
f_0 = f_{0,1} + f_{0,2} + \cdots + f_{0,j},
$$

for some  $f_{0,m}\in L^{p',\infty}(\mathcal X),$  satisfying

$$
\mathscr{L} f_{0,m} = \gamma(\alpha_m + i\delta_{p'}) \ f_{0,m}, \quad \text{for all } m = 1,\ldots,j,
$$

where  $-\tau/2 < \alpha_m \leq \tau/2$  are distinct and  $|\Psi \circ \gamma(\alpha_m + i\delta_{p'})| = |A|.$ 

Math. Z., 2025)<br>
n  $S_p(\mathcal{X})$  with symbol  $\Psi \circ \gamma$  such<br>
hat  $\{f_{-k}\}_{k \in \mathbb{Z}_+}$  is a bi-infinite<br>  $= A f_{-k+1}$ , for all  $k \in \mathbb{N}$ .<br>  $|w| = |A|\}$  at finitely many distinct<br>  $\cdots + f_{0,j}$ ,<br>
for all  $m = 1, \ldots, j$ ,<br>  $\Psi \circ \gamma(\alpha_m$ For  $1 \leq p \leq 2$ . Let  $\Psi(\mathscr{L})$  be a multiplier on  $\mathscr{S}_p(\mathscr{X})$  with symbol  $\Psi \circ \gamma$  such that  $\Psi \circ \gamma(z) \neq 0$  for all  $z \in S_p$ . Suppose that  $\{f_{-k}\}_{k \in \mathbb{Z}_+}$  is a bi-infinite sequence of functions on  $\mathcal X$  satisfying

$$
||f_{-k}||_{L^{p'},\infty(\mathcal{X})} \leq M \text{ and } \Psi(\mathcal{L})f_{-k} = A \ f_{-k+1}, \text{ for all } k \in \mathbb{N}.
$$

Assume further that

- (a)  $|A| = \max\{|\Psi \circ \gamma(z)| : z \in S_p\}.$
- (b) The range of  $\Psi \circ \gamma$  intersects  $\{w \in \mathbb{C} : |w| = |A|\}$  at finitely many distinct points.

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# <span id="page-104-0"></span>[Notable Consequences](#page-104-0)

I

Ext  $\chi_{S(o,n)}$  denote the indicator function of the sphere  $S(o, n)$ .



# Spherical averages on  $X$

- **I** Let  $\chi_{S(o,n)}$  denote the indicator function of the sphere  $S(o,n)$ .
- **The spherical average of a function f over**  $S(x, n)$  **is given by**

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- $\bullet$  Observation :  $\mathscr{S}_0 f = f$  and  $\mathscr{S}_1 f = f \mathscr{L} f$ .
- $\bullet$  **Fact :** For *n* ≥ 2,

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# Spherical averages on  $X$

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- $\bullet$  Observation :  $\mathscr{S}_0 f = f$  and  $\mathscr{S}_1 f = f \mathscr{L} f$ .
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**Therefore,**  $\mathscr{S}_n = P_n(\mathscr{L})$ , where  $P_n$  is a polynomial of degree n.

 $\bullet$  Symbol :  $z \mapsto \phi_z(n)$ .

## Strichartz's theorem for spherical averages on  $\mathcal X$

- **The maximum modulus of**  $z \mapsto \phi_z(n)$  is  $\phi_{i\delta_{p'}}(n) = -\phi_{\tau/2+i\delta_{p'}}(n)$ .
- **Attained at**  $z_1 = i\delta_{p'}$  **and**  $z_2 = \tau/2 + i\delta_{p'}$ **.**
- $\bullet$  The range of  $z \mapsto \phi_z(n)$  contains zero.

### **Corollary**

s on  $\mathscr{X}$ <br>  $\phi_{i\delta_{\rho'}}(n) = -\phi_{\tau/2+i\delta_{\rho'}}(n).$ <br>  $\delta_{\rho'}.$ <br>
bi-infinite sequence of functions<br>
A f<sub>k+1</sub>, for all  $k \in \mathbb{Z}$ ,<br>
f<sub>0</sub> can be uniquely written as<br>  $\delta_{0,2}$ ,<br>  $\phi_{2,2} = \gamma(\tau/2 + i\delta_{\rho'})$  f<sub>0,2</sub>. Fix  $n \in \mathbb{N}$ . For  $1 \leq p < 2$ , let  $\{f_k\}_{k \in \mathbb{Z}}$  be a bi-infinite sequence of functions on  $\mathscr X$  satisfying

$$
||f_k||_{L^{p'},\infty(\mathcal{X})} \leq M \text{ and } \mathscr{S}_n f_k = A \ f_{k+1}, \text{ for all } k \in \mathbb{Z},
$$

where  $A\in\mathbb{C}$  satisfies  $|A|=\phi_{i\delta_{p'}}(n).$  Then  $f_0$  can be uniquely written as

$$
\mathit{f}_0 = \mathit{f}_{0,1} + \mathit{f}_{0,2},
$$

for some  $f_{0,1}, f_{0,2} \in L^{p',\infty}(\mathcal X)$  satisfying

$$
\mathscr{L} f_{0,1} = \gamma \big( i \delta_{\rho'} \big) \ f_{0,1} \quad \text{and} \quad \mathscr{L} f_{0,2} = \gamma \big( \tau/2 + i \delta_{\rho'} \big) \ f_{0,2}.
$$

## The heat operator on  $\mathscr X$

For  $\xi \in \mathbb{C}^{\times}$ , the complex-time heat operator  $\mathcal{H}_{\xi}$  is defined by

$$
\mathscr{H}_{\xi}f(x)=f*h_{\xi}(x),
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where  $h_{\varepsilon}$  denotes the heat kernel on  $\mathcal{X}.$ 



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**Example 1** Symbol :  $\widehat{h}_{\xi}(z) = e^{\xi \gamma(z)}$ .



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18.2. Let 
$$
\mathcal{C}^{\times}
$$
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\n
$$
\mathcal{H}_{\xi}f(x) = f * h_{\xi}(x),
$$
\n
$$
\xi
$$
 denotes the heat Kernel on  $\mathcal{X}$ :\n
$$
\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}.
$$
\ng.e.

\n
$$
\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}.
$$
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\n
$$
\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}.
$$
\nhence,  $\hat{h}_{\xi}(z) = (1 - \gamma(i\delta_{\rho'})) \cdot ((\Re \xi)^2 + \tanh^2(\delta_{\rho'} \log q)(\Im \xi)^2)^{1/2}.$ 

\ng.e.

\n
$$
\Phi_{\rho}(\xi) = (1 - \gamma(i\delta_{\rho'})) \cdot ((\Re \xi)^2 + \tanh^2(\delta_{\rho'} \log q)(\Im \xi)^2)^{1/2}.
$$

### The heat operator on  $\mathscr X$

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1. (c) 
$$
\mathscr{X}
$$

\n2. (d)  $\mathscr{X}_{\xi}$ 

\n3. (e)  $\mathscr{W}_{\xi}$ 

\n4. (f)  $\mathscr{W}_{\xi}$ 

\n5. (g)  $\mathscr{W}_{\xi}$ 

\n6. (h)  $\mathscr{W}_{\xi}$ 

\n7. (i)  $\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}$ 

\n8. (g)  $\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}$ 

\n9. (h)  $\hat{h}_{\xi}(z) = e^{\xi \gamma(z)}$ 

\n1. (i)  $\hat{h}_{\xi}(z) = (1 - \gamma(i\delta_{\rho'})) \cdot ((\Re \xi)^2 + \tanh^2(\delta_{\rho'} \log q)(\Im \xi)^2)^{1/2}$ 

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\n2. (i)  $\hat{h}_{\xi}(z) = \exp\{\Re \xi + \Phi_{\rho}(\xi)\}$ 

\n3. (ii)  $\hat{h}_{\xi}(z) = \exp\{\Re \xi - \Phi_{\rho}(\xi)\}$ 

- **E** The maximum modulus :  $exp{\Re\xi + \Phi_p(\xi)}$ .
- **E** The minimum modulus : exp{ $\Re \xi \Phi_p(\xi)$ }.

 $\bullet$  Let  $\beta_i$ ,  $j = 1, 2$ , denote the unique points in  $(-\tau/2, \tau/2]$  satisfying

 $\Phi_{p}(\xi) \cos \beta_{j} = (-1)^{j} \Re \xi \cdot (1 - \gamma(i \delta_{p'})), \ \Phi_{p}(\xi) \sin \beta_{j} = (-1)^{j} \Im \xi \cdot \gamma(\tau/4 + i \delta_{p'}).$ 

**•** Maximum and minumim modulus are attained at  $z_1 = \beta_1 + i\delta_{p'}$  and  $z_2 = \beta_2 + i\delta_{p'}$ , respectively.

### **Corollary**

Fix  $\xi \in \mathbb{C}^\times$ . For  $1 \leq p < 2$ , let  $\{f_k\}_{k \in \mathbb{Z}}$  be a bi-infinite sequence of functions on  $\mathscr X$  such that  $||f_k||_{L^{p',\infty}(\mathscr X)}\leq M$  for all  $k\in\mathbb Z$ .

- **r on** *X*<br>
ts in  $(-\tau/2, \tau/2]$  satisfying<br>  $\rho(\xi) \sin \beta_j = (-1)^j \Im \xi \cdot \gamma(\tau/4 + i\delta_{\rho'})$ .<br>
ttained at  $z_1 = \beta_1 + i\delta_{\rho'}$  and<br>
a bi-infinite sequence of functions<br>  $\epsilon \mathbb{Z}$ .<br>
re *A* ∈ *C* satisfies  $|A| = \exp{\Re \xi} + \beta_1 + i\delta_{\rho'}$  a (a) If  $\mathcal{H}_{\xi}f_{-k} = A f_{-k+1}$  for all  $k \in \mathbb{N}$ , where  $A \in \mathbb{C}$  satisfies  $|A| = \exp{\Re \xi + \pi}$  $\Phi_{p}(\xi)\}$ , then  $\mathscr{L}$  fo  $=\gamma(z_1)$  fo, where  $z_1=\beta_1+i\delta_{p'}$  and  $\beta_1$  is as above.
- (b) If  $\mathcal{H}_{\xi}f_k = A f_{k+1}$  for all  $k \in \mathbb{Z}_+$ , where  $A \in \mathbb{C}$  satisfies  $|A| = \exp{\Re \xi \pi}$  $\Phi_{p}(\xi)\}\$ , then  $\mathscr{L} f_0 = \gamma(z_2)f_0$ , where  $z_2 = \beta_2 + i\delta_{p'}$  and  $\beta_2$  is as above.



S.

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# I Thank You !