A theorem of Strichartz for multipliers on homogeneous trees

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(Based on a joint work with R. P. Sarkar)

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Department of Mathematics

Indian Institute of Science

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Overview of the Talk

A Theorem of Roe and Strichartz on $\mathbb{R}^n$

A brief history

- **J. Roe, 1980 :** Let $\{f_k\}_{k\in\mathbb{Z}}$ be a doubly infinite sequence of functions on $\mathbb{R}$ such that

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- If $\Delta_{\mathbb{R}^n} f_k = A\ f_{k+1}$ for some $A \in \mathbb{C}^\times$, then $\Delta_{\mathbb{R}^n} f_0 = -|A| f_0$.

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Question

Does a precise analogue of Strichartz's theorem apply to the combinatorial Laplacian $\mathscr{L}$ on a homogeneous tree $\mathscr{X}$?

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- The boundary Ω is identified with the set of all infinite geodesic rays starting at o.

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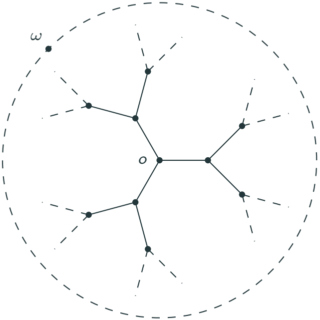

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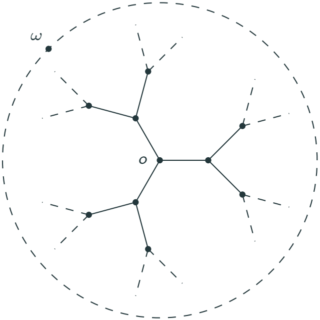

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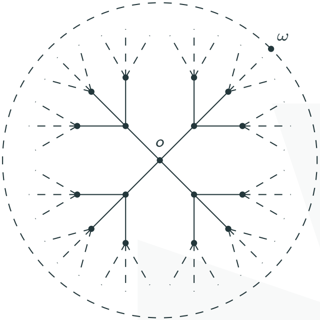

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$$\mathscr{L}u(x) = u(x) - \frac{1}{q+1} \sum_{y:d(x,y)=1} u(y).$$

Strichartz's Theorem for $\mathscr{L}$ on $\mathscr{X}$

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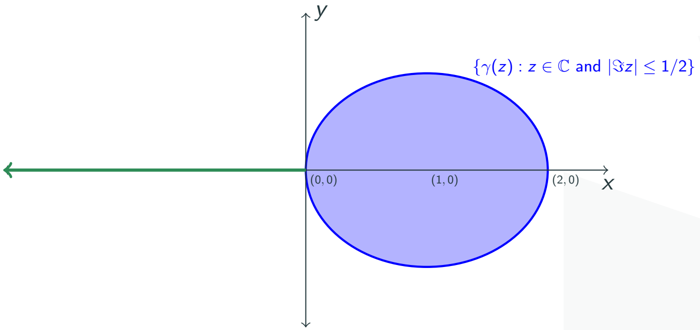
- Unlike the L^∞-point spectrum of $\Delta_{\mathbb{R}^n}$ which is the one-dimensional interval $(-\infty, 0]$, the L^∞-point spectrum of $\mathscr{L}$ is an elliptic region in the complex plane centered around the point 1.



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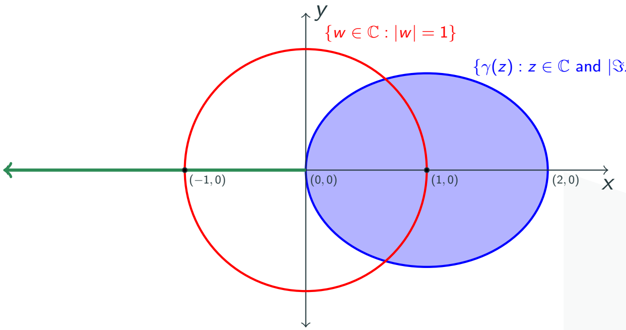

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- Choose two points z_1, z_2 in $\{z \in \mathbb{C} : |\Im z| \leq 1/2\}$ such that

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- Therefore $\{f_k\}_{k \in \mathbb{Z}}$ satisfies all the hypothesis of Strichartz's theorem.
- However, f_0 fails to be an eigenfunction of $\mathscr{L}$.

Size estimates of ϕ_z

- Notations for today : Let $1 < p \leq 2$. Then
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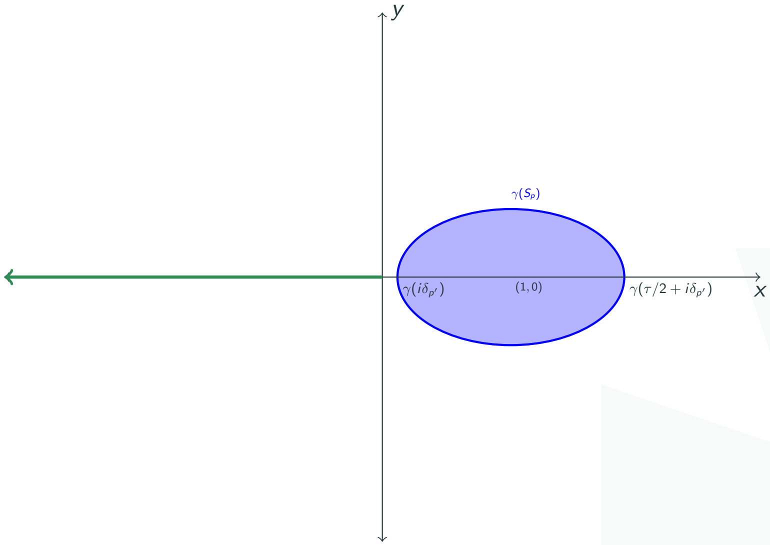
$L^{p',\infty}$-point spectrum of $\mathscr{L}$, for $1 \leq p \leq 2$



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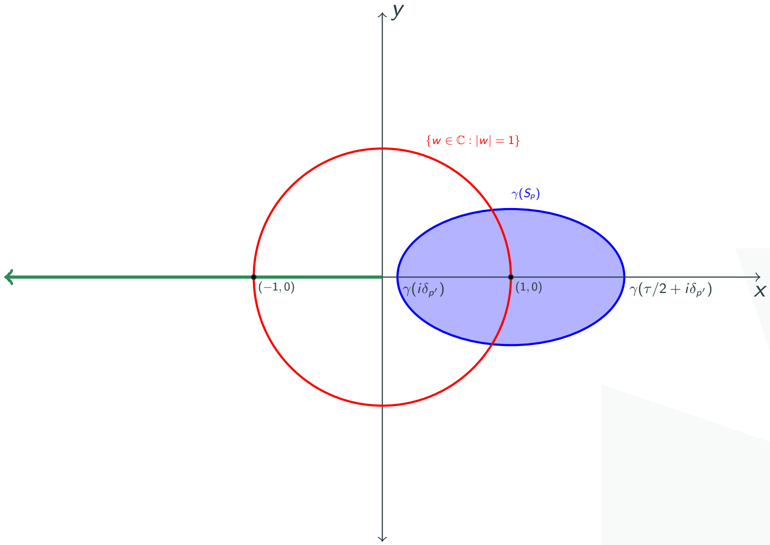

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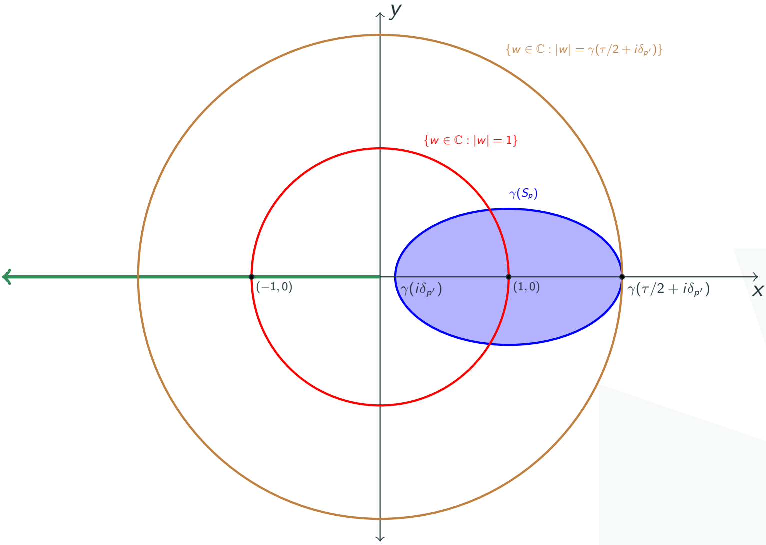

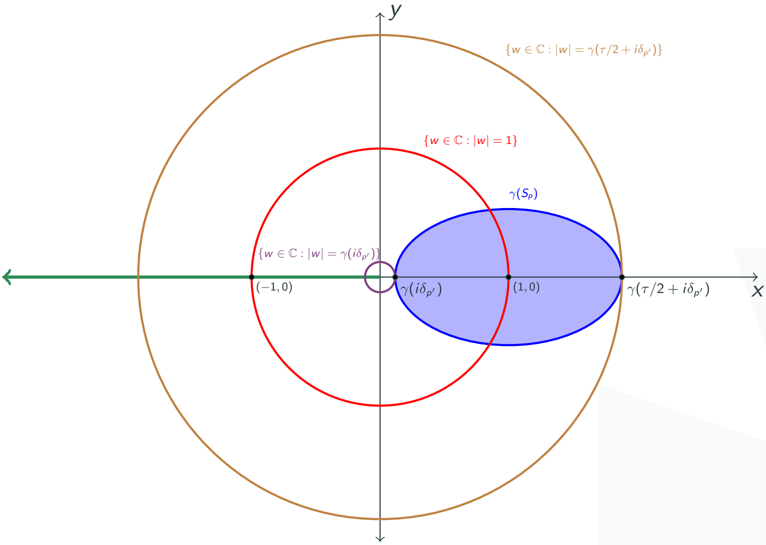
y
x
$\{w \in \mathbb{C} : |w| = \gamma(\tau/2 + i\delta_{p'})\}$
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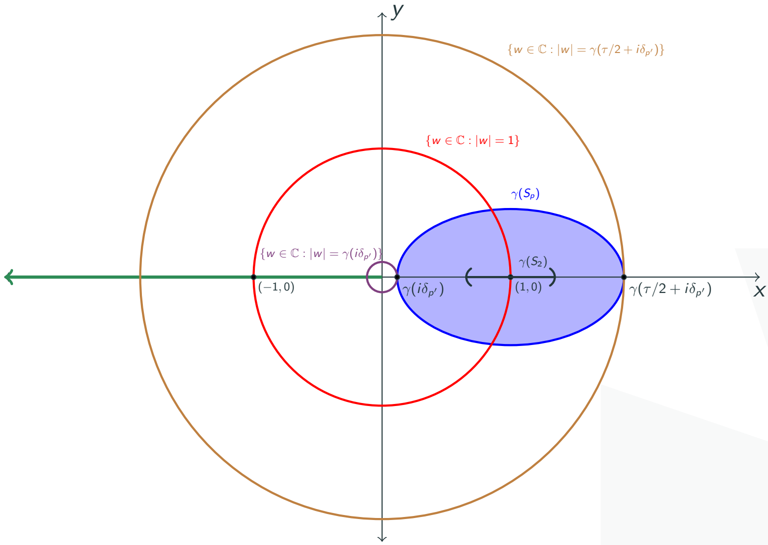

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- We shall specifically focus on extending the above results for multipliers when $1 \leq p < 2$.

Strichartz's Theorem for Multipliers

Fourier transforms on $\mathscr{X}$

- The spherical transform $\widehat{f}$ of a finitely supported radial function f on $\mathscr{X}$ is defined by the formula

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- If f is radial, then $\widetilde{f}(z, \omega) = \widehat{f}(z)$, for all $\omega \in \Omega$.

L^p-Schwartz spaces on $\mathscr{X}$, $1 \leq p \leq 2$

- Schwartz spaces $\mathscr{S}_p(\mathscr{X})$: Space of all functions ϕ on $\mathscr{X}$ for which

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Definition

Let m be an even, τ-periodic, bounded measurable function on $\mathbb{R}$. An operator Θ defined as

$$\Theta f(x) = c_{\mathscr{X}} \int_{\mathbb{T}} \int_{\Omega} m(z) \, \widetilde{f}(z, \omega) \, p^{1/2 - iz}(x, \omega) \, |c(z)|^{-2} \, d\nu(\omega) \, dz,$$

is said to be a multiplier on $\mathscr{S}_p(\mathscr{X})$ with symbol $m(z)$ if, for every semi-norm $\nu_{p,m_2}(\cdot)$ of $\mathscr{S}_p(\mathscr{X})$, there exists a semi-norm $\nu_{p,m_1}(\cdot)$ of $\mathscr{S}_p(\mathscr{X})$ and a constant $C_{m_1,m_2} > 0$ such that

$$\nu_{p,m_2}(\Theta f) \le C_{m_1,m_2} \, \nu_{p,m_1}(f), \quad \text{for all } f \in \mathscr{S}_p(\mathscr{X}).$$

Characterization of multipliers on the Schwartz spaces

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Proposition (S. K. Rano and R. P. Sarkar ; Math. Z. , 2025)

Let $1 \leq p < 2$. Then the following are equivalent.

(a) The operator Θ is a multiplier on $\mathscr{S}_p(\mathscr{X})$ with symbol $m(z)$.

(b) m is in $\mathscr{H}(S_p)$.

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- The ball averaging operators $\mathscr{B}_n$ is a multiplier on $\mathcal{S}_p(\mathscr{X})$ with symbol $\psi_z(n)$, where

$$\psi_z(n) = \frac{1}{\#B(o,n)} \sum_{j=0}^{n} \#S(o,j)\, \phi_z(j), \quad \text{for all } n \in \mathbb{Z}_+.$$

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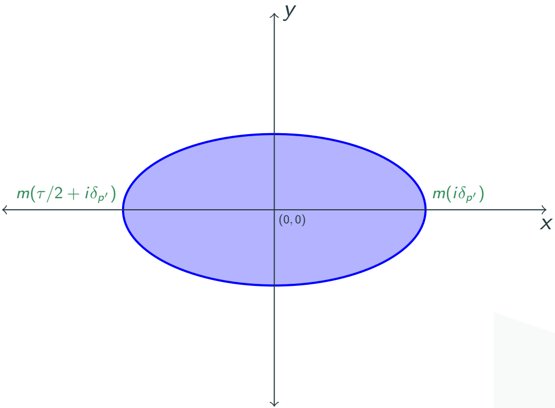
- **Difficulty :** The range of m may intersect $\{w \in \mathbb{C} : |w| = |A|\}$ at more than one point.

Pictorial representation

- Multiplier : $I - \mathscr{L}$. Symbol : $m(z) = 1 - \gamma(z)$.



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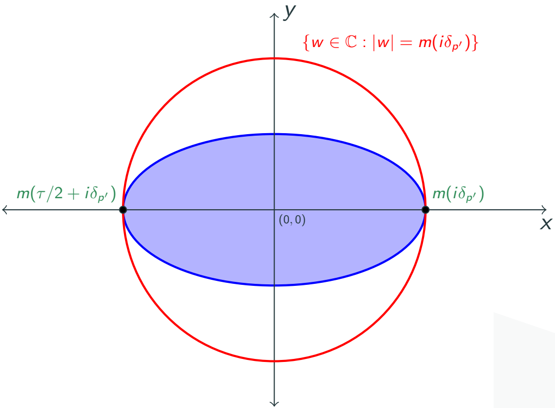

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Strichartz's theorem for multipliers I

Theorem (S. K. Rano and R. P. Sarkar ; Math. Z. , 2025)

Let $1 \leq p < 2$. *Let* Θ *be a multiplier on* $\mathcal{S}_p(\mathcal{X})$ *with symbol* $m(z)$ *satisfying* $m(z) \neq 0$ *for some* $z \in S_p$. *Suppose that* $\{f_k\}_{k \in \mathbb{Z}}$ *is a bi-infinite sequence of functions on* $\mathcal{X}$ *satisfying*

$$\|f_k\|_{L^{p',\infty}(\mathcal{X})} \leq M \quad \text{and} \quad \Theta f_k = A\, f_{k+1}, \quad \text{for all } k \in \mathbb{Z}.$$

Assume further that

(a) $|A| = \max\{|m(z)| : z \in S_p\}$.

(b) The range of m *intersects* $\{w \in \mathbb{C} : |w| = |A|\}$ *at finitely many distinct points* $A_1, \ldots, A_j$.

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Strichartz's theorem for multipliers II

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Strichartz's theorem for multipliers III

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Let $1 \leq p < 2$. *Let* Θ *be a multiplier on* $\mathscr{S}_p(\mathscr{X})$ *associated with symbol* $m(z)$ *satisfying* $m(z) \neq 0$ *for all* $z \in S_p$. *Suppose that* $\{f_{-k}\}_{k \in \mathbb{Z}_+}$ *is a bi-infinite sequence of functions on* $\mathscr{X}$ *satisfying*

$$\|f_{-k}\|_{L^{p',\infty}(\mathscr{X})} \leq M \quad \text{and} \quad \Theta f_{-k} = A\, f_{-k+1}, \quad \text{for all } k \in \mathbb{N}.$$

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- YES ! If the multipliers are functions of the Laplacian.
- Let Ψ be a nonconstant holomorphic function defined on a connected open set containing $\gamma(S_p)$.
- Then, $\Psi \circ \gamma$ is in $\mathscr{H}(S_p)$.
- Hence, $\Psi \circ \gamma$ corresponds to a multiplier on $\mathcal{S}_p(\mathscr{X})$, which will be denoted by $\Psi(\mathscr{L})$.

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- Hence, $\Psi \circ \gamma$ corresponds to a multiplier on $\mathcal{S}_p(\mathscr{X})$, which will be denoted by $\Psi(\mathscr{L})$.
- **Key examples :** Polynomials of $\mathscr{L}$, the spherical and the ball averages on $\mathscr{X}$, the heat operator on $\mathscr{X}$.

Strichartz's theorem for $\Psi(\mathscr{L})$ I

Theorem (S. K. Rano and R. P. Sarkar ; Math. Z. , 2025)

For $1 \leq p < 2$. *Let* $\Psi(\mathscr{L})$ *be a multiplier on* $\mathscr{S}_p(\mathscr{X})$ *associated with the symbol* $\Psi \circ \gamma$. *Suppose that* $\{f_k\}_{k\in\mathbb{Z}}$ *is a bi-infinite sequence of functions on* $\mathscr{X}$ *satisfying*

$$\|f_k\|_{L^{p',\infty}(\mathscr{X})} \leq M \quad \text{and} \quad \Psi(\mathscr{L})f_k = A\ f_{k+1}, \quad \text{for all } k \in \mathbb{Z}.$$

Assume further that

(a) $|A| = \max\{|\Psi \circ \gamma(z)| : z \in S_p\}$.

(b) The range of $\Psi \circ \gamma$ *intersects* $\{w \in \mathbb{C} : |w| = |A|\}$ *at finitely many distinct points.*

Then f_0 *can be uniquely written as*

$$f_0 = f_{0,1} + f_{0,2} + \cdots + f_{0,j},$$

for some $f_{0,m} \in L^{p',\infty}(\mathscr{X})$, *satisfying*

$$\mathscr{L} f_{0,m} = \gamma(\alpha_m + i\delta_{p'})\ f_{0,m}, \quad \text{for all } m = 1, \ldots, j,$$

where $-\tau/2 < \alpha_m \leq \tau/2$ *are distinct and* $|\Psi \circ \gamma(\alpha_m + i\delta_{p'})| = |A|$.

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(a) $|A| = \min\{|\Psi \circ \gamma(z)| : z \in S_p\}$.

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For $1 \leq p < 2$. *Let* $\Psi(\mathscr{L})$ *be a multiplier on* $\mathscr{S}_p(\mathscr{X})$ *with symbol* $\Psi \circ \gamma$ *such that* $\Psi \circ \gamma(z) \neq 0$ *for all* $z \in S_p$. *Suppose that* $\{f_{-k}\}_{k \in \mathbb{Z}_+}$ *is a bi-infinite sequence of functions on* $\mathscr{X}$ *satisfying*

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Assume further that

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Notable Consequences

Spherical averages on $\mathscr{X}$

- Let $\chi_{S(o,n)}$ denote the indicator function of the sphere $S(o, n)$.

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- The spherical average of a function f over $S(x,n)$ is given by

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- **Fact :** For $n \geq 2$,

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- Therefore, $\mathscr{S}_n = P_n(\mathscr{L})$, where P_n is a polynomial of degree n.
- Symbol : $z \mapsto \phi_z(n)$.

Strichartz's theorem for spherical averages on $\mathscr{X}$

- The maximum modulus of $z \mapsto \phi_z(n)$ is $\phi_{i\delta_{p'}}(n) = -\phi_{\tau/2+i\delta_{p'}}(n)$.
- Attained at $z_1 = i\delta_{p'}$ and $z_2 = \tau/2 + i\delta_{p'}$.
- The range of $z \mapsto \phi_z(n)$ contains zero.

Corollary

Fix $n \in \mathbb{N}$. For $1 \leq p < 2$, let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of functions on $\mathscr{X}$ satisfying

$$\|f_k\|_{L^{p',\infty}(\mathscr{X})} \leq M \quad \text{and} \quad \mathscr{S}_n f_k = A\, f_{k+1}, \quad \text{for all } k \in \mathbb{Z},$$

where $A \in \mathbb{C}$ satisfies $|A| = \phi_{i\delta_{p'}}(n)$. Then f_0 can be uniquely written as

$$f_0 = f_{0,1} + f_{0,2},$$

for some $f_{0,1}, f_{0,2} \in L^{p',\infty}(\mathscr{X})$ satisfying

$$\mathscr{L} f_{0,1} = \gamma(i\delta_{p'})\, f_{0,1} \quad \text{and} \quad \mathscr{L} f_{0,2} = \gamma(\tau/2 + i\delta_{p'})\, f_{0,2}.$$

The heat operator on $\mathscr{X}$

- For $\xi \in \mathbb{C}^\times$, the complex-time heat operator $\mathscr{H}_\xi$ is defined by

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- For $1 \leq p < 2$, we define

$$\Phi_p(\xi) = (1 - \gamma(i\delta_{p'})) \cdot ((\Re\xi)^2 + \tanh^2(\delta_{p'} \log q)(\Im\xi)^2)^{1/2}.$$

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$$\Phi_p(\xi) = (1 - \gamma(i\delta_{p'})) \cdot ((\Re\xi)^2 + \tanh^2(\delta_{p'} \log q)(\Im\xi)^2)^{1/2}.$$

- The maximum modulus : $\exp\{\Re\xi + \Phi_p(\xi)\}$.

- The minimum modulus : $\exp\{\Re\xi - \Phi_p(\xi)\}$.

- Let β_j, $j = 1, 2$, denote the unique points in $(-\tau/2, \tau/2]$ satisfying

  $$\Phi_p(\xi)\cos\beta_j = (-1)^j \Re\xi\cdot(1-\gamma(i\delta_{p'})),\ \Phi_p(\xi)\sin\beta_j = (-1)^j \Im\xi\cdot\gamma(\tau/4+i\delta_{p'}).$$

- Maximum and minumim modulus are attained at $z_1 = \beta_1 + i\delta_{p'}$ and $z_2 = \beta_2 + i\delta_{p'}$, respectively.

Corollary

Fix $\xi \in \mathbb{C}^\times$. For $1 \leq p < 2$, let $\{f_k\}_{k\in\mathbb{Z}}$ be a bi-infinite sequence of functions on $\mathscr{X}$ such that $\|f_k\|_{L^{p',\infty}(\mathscr{X})} \leq M$ for all $k \in \mathbb{Z}$.

(a) If $\mathscr{H}_\xi f_{-k} = A\ f_{-k+1}$ for all $k \in \mathbb{N}$, where $A \in \mathbb{C}$ satisfies $|A| = \exp\{\Re\xi + \Phi_p(\xi)\}$, then $\mathscr{L} f_0 = \gamma(z_1) f_0$, where $z_1 = \beta_1 + i\delta_{p'}$ and β_1 is as above.

(b) If $\mathscr{H}_\xi f_k = A\ f_{k+1}$ for all $k \in \mathbb{Z}_+$, where $A \in \mathbb{C}$ satisfies $|A| = \exp\{\Re\xi - \Phi_p(\xi)\}$, then $\mathscr{L} f_0 = \gamma(z_2) f_0$, where $z_2 = \beta_2 + i\delta_{p'}$ and β_2 is as above.

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Thank You !