Politecnico di Torino
Dipartimento di Scienze Matematiche "G. L. Lagrange"
DISMA

Sharp L^p estimates for the sub-Riemannian wave equation

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joint work with

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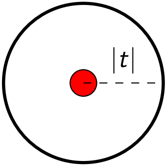

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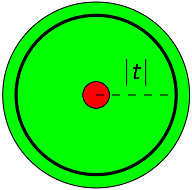

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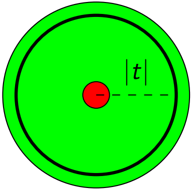

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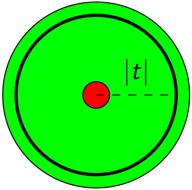

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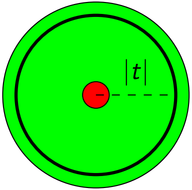

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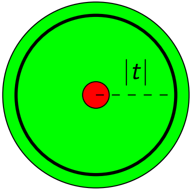

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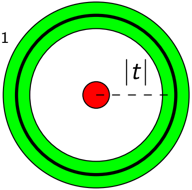

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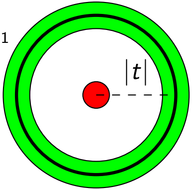

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note: $Q > d$ for subelliptic, nonelliptic $\mathscr{L}$

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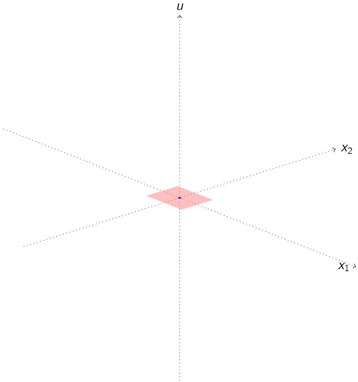
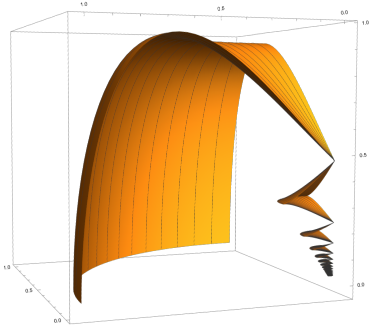
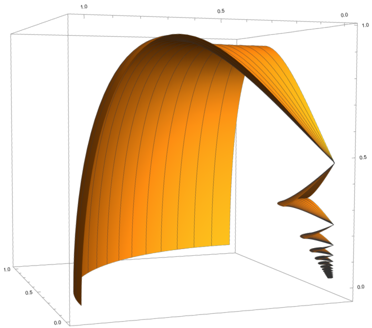




isotropic Heisenberg group H_1



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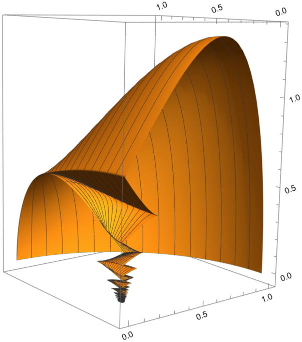
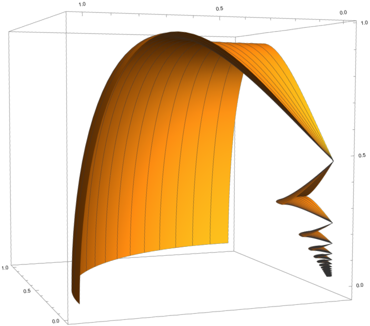


anisotropic Heisenberg group H_2



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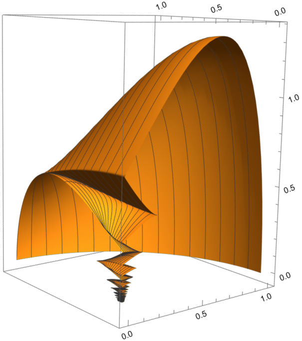

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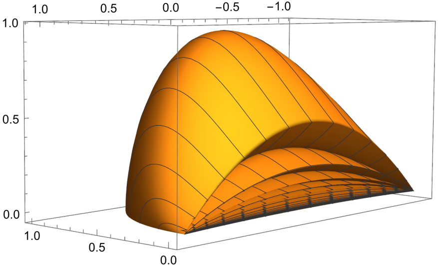

direct product $H_1 \times \mathbb{R}$

(Almost) sharp Miyachi–Peral estimates

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Theorem (M. & Müller, arXiv:2406.04315)

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$\varsigma_{\mathrm{MP}}(\mathscr{L}) = (d-1)/2$ for $\mathscr{L}$ sub-Laplacian on a Métivier group, or on a direct product of Heisenberg groups and Euclidean spaces.

The Seeger–Sogge–Stein method: the Euclidean case

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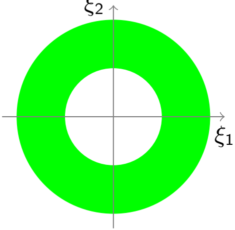

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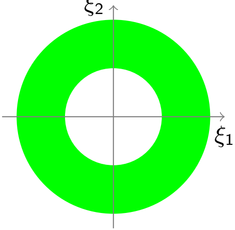

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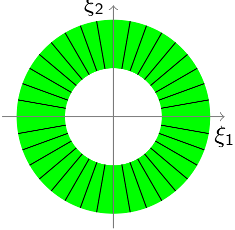

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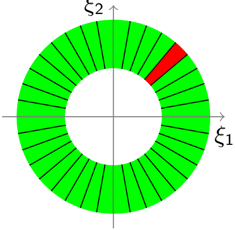

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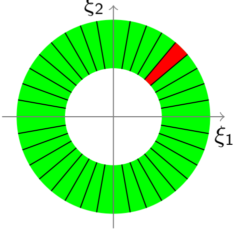

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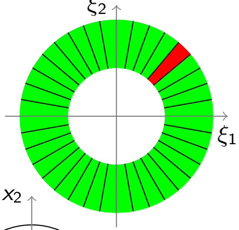

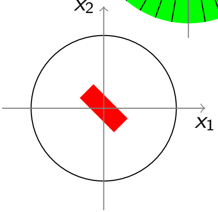

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case $\mathscr{L} = -\Delta$ Laplacian on $\mathbb{R}^d$

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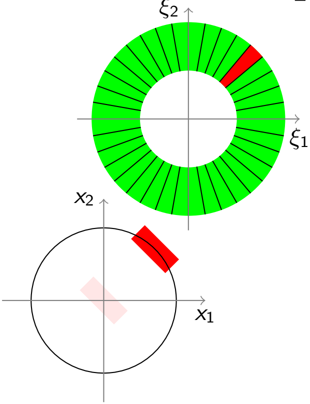

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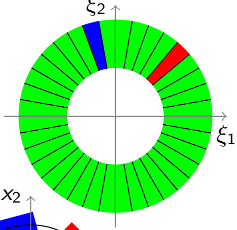

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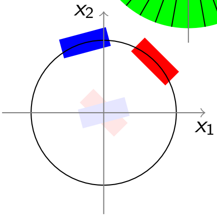

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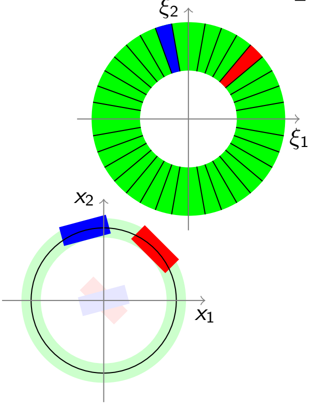

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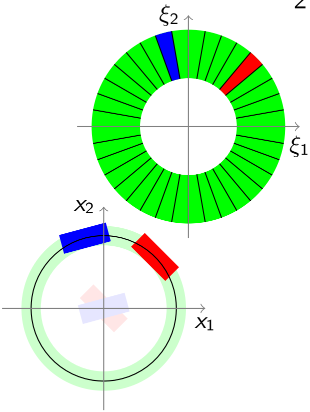

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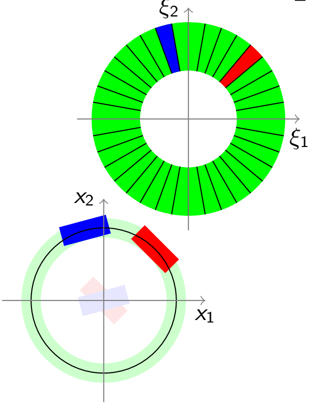

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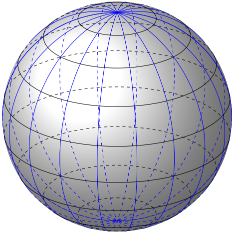
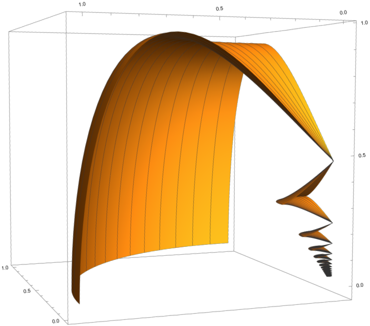
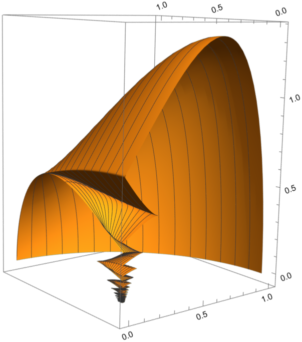


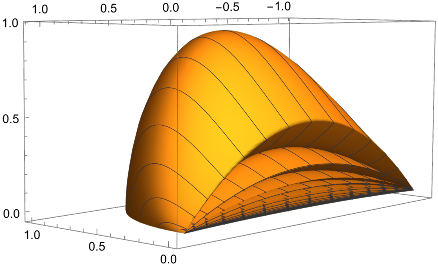
So we can proceed as before with the Seeger–Sogge–Stein argument (at least on a compact manifold).



isotropic Heisenberg group H_1



anisotropic Heisenberg group H_2



direct product $H_1 \times \mathbb{R}$

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- problem: how to relate frequency and spectral localisations?

Spectral vs frequency localisations

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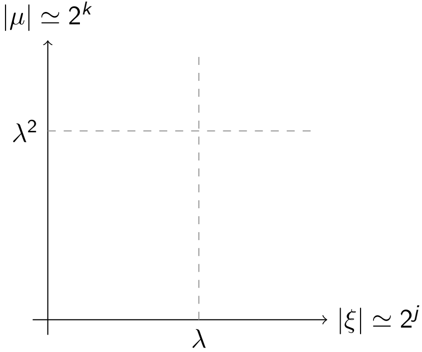

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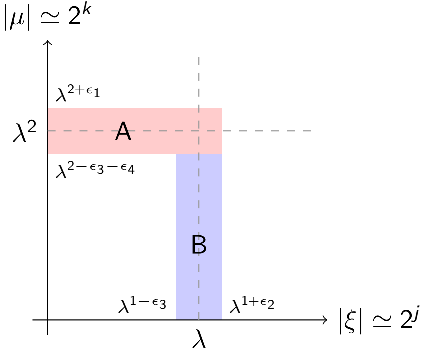

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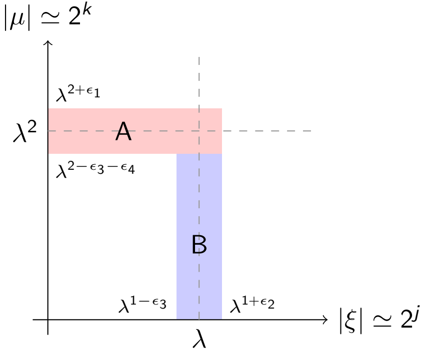

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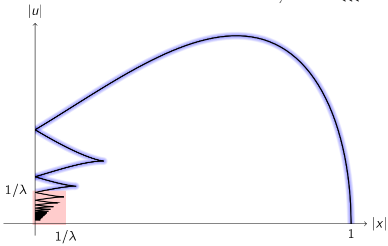

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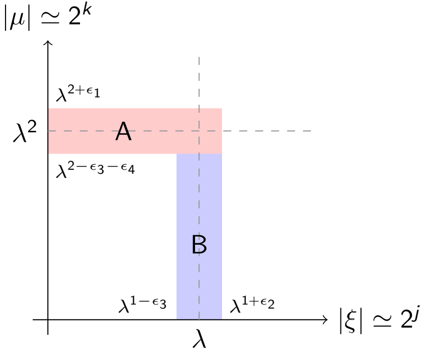

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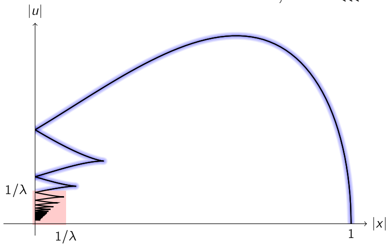

- for region A: use first-layer weighted Plancherel estimates to exploit further spatial localisation (ess. no propagation in x, as $|\xi|/|\mu| \lessapprox \lambda^{-1}$)

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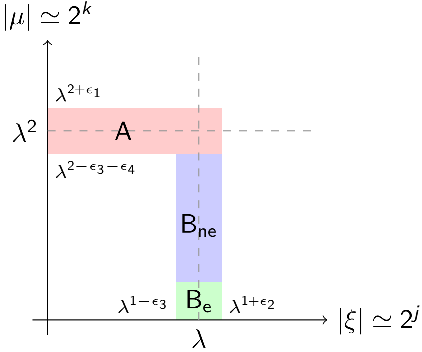

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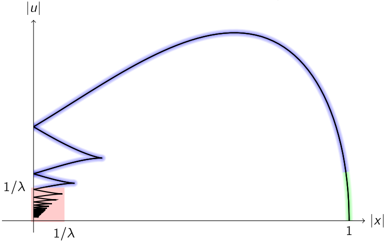
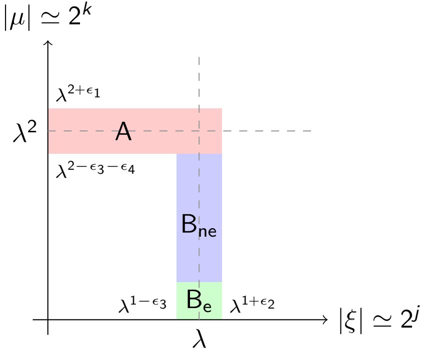
  - A: $2^j \lessapprox \lambda, \quad 2^k \approx \lambda^2$
  - B: $2^j \approx \lambda, \quad 2^k \lll \lambda^2$



- for region A: use first-layer weighted Plancherel estimates to exploit further spatial localisation (ess. no propagation in x, as $|\xi|/|\mu| \lessapprox \lambda^{-1}$)
- for region B: further split into
  - elliptic: $2^k \ll 2^j$
  - nonelliptic: $2^j \lesssim 2^k \lll \lambda^2$

Spectral vs frequency localisations

- localised wave propagator: $\mathbf{1}_{\overline{B}(0,2)}\chi_1(\sqrt{\mathscr{L}}/\lambda)\cos(\sqrt{\mathscr{L}})\delta_0$
- frequency decomposition $\delta_0 = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\chi_1(2^{-j}|D_x|)\chi_1(2^{-k}|D_u|)\delta_0$



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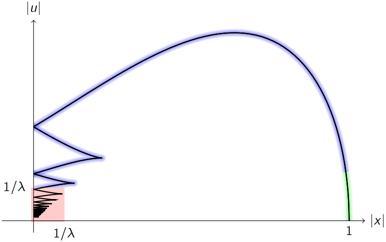

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- elliptic region: dealt with classical FIO approach (here $\mathcal{H}(\boldsymbol{x},\boldsymbol{\xi}) \simeq |\boldsymbol{\xi}|^2$) (cf. [M. & Müller & Nicolussi Golo '23] + [Seeger & Sogge & Stein '91])

Nonelliptic region: parabolic scaling

- must prove

  $$\|\mathbf{1}_{\overline{B}(0,4)} \cos(\sqrt{\mathscr{L}})\chi_1(2^{-j}|D_x|)\chi_1(2^{-k}|D_u|)\delta_0\|_1 \lessapprox \lambda^{(d-1)/2}$$

  for

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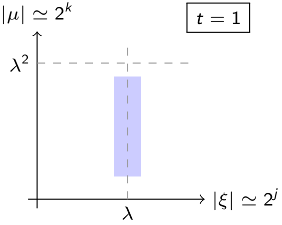

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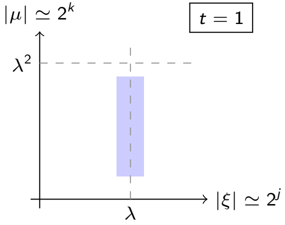

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  $$(\xi, \mu) \mapsto (2^\ell \xi, 2^{2\ell}\mu), \quad (x, u) \mapsto (2^{-\ell}x, 2^{-2\ell}u)$$

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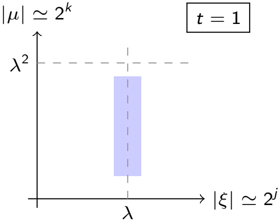

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  (with $\ell = k - j$, $m = 2j - k$) reduce to

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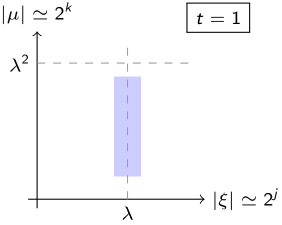

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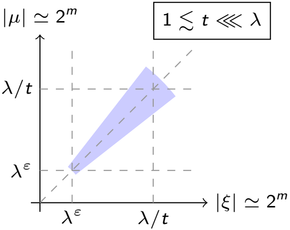
- effectively we are reduced to the frequency region

  $$|\xi| \simeq |\mu| \simeq 2^m,$$

  but we need estimates for <u>large time</u> $t = 2^\ell$ (!)



Large-time FIO parametrix via complex phase
(à la [Laptev–Safarov–Vassiliev '94])

- represent
$$\cos(t\sqrt{\mathscr{L}})\chi_1(2^{-m}|D_x|)\chi_1(2^{-m}|D_u|)\delta_0 = \frac{Q_t^m + Q_{-t}^m}{2} + \text{l.o.t.}$$
$$Q_t^m(\boldsymbol{x}) = \int e^{i\phi(t,\boldsymbol{x},\boldsymbol{\xi})}\, q_m(t,\boldsymbol{\xi})\, \mathfrak{d}_\phi(t,\boldsymbol{x},\boldsymbol{\xi})\, d\boldsymbol{\xi}$$

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- for $|t| \gg 1$: possible blow-up of $\boldsymbol{\xi}$-derivatives of ϕ (!)

The large time regime - 1

- phase $\phi(t, \boldsymbol{x}, \boldsymbol{\xi}) = (\boldsymbol{x} - \boldsymbol{x}^t) \cdot \boldsymbol{\xi}^t + i\frac{|\mu|}{4}|x - x^t|^2$

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where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $\theta = \dfrac{t\mu}{2|\xi|}$

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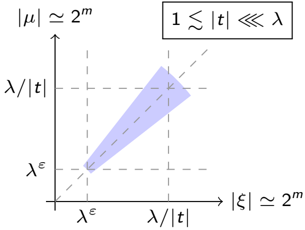

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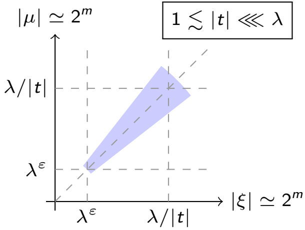

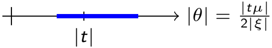

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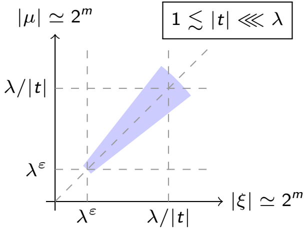

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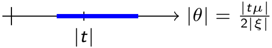

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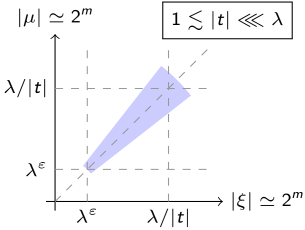

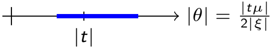

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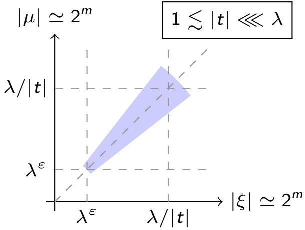

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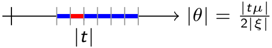

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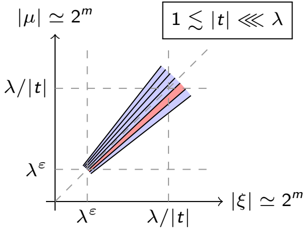

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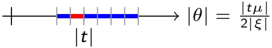

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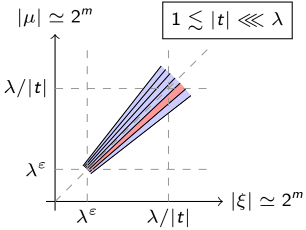

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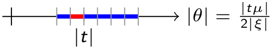

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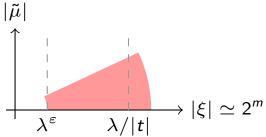

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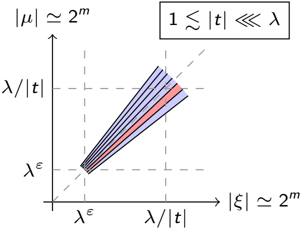

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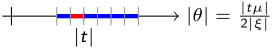

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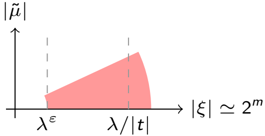

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  - now $(\xi, \tilde{\mu})$-derivatives of $\sin(k + \tilde{\theta})$ and $\cos(k + \tilde{\theta})$ are no longer problematic!

The large time regime - 2

- via this decomposition and change of variables, we have

$$Q_t^m(\boldsymbol{x}) = |t|^{-1/2} \sum_{|k| \simeq |t|} Q_t^{m,k}(\boldsymbol{x})$$

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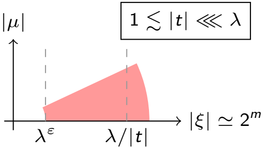

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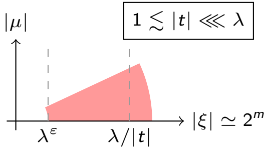

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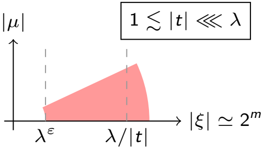

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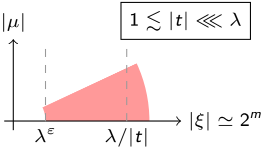

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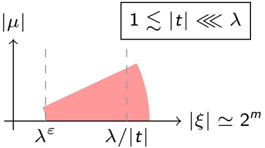

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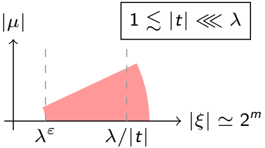

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as desired

Heisenberg-type and Métivier groups

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  - as $d_2 < d_1$ for Métivier groups, the additional blow-up can be absorbed:

$$|t|^{d_2-1}|t|^{1/2}2^{m(d-1)/2} = 2^{\ell(d_2-1/2)}2^{m(d-1)/2} \lesssim \lambda^{(d-1)/2}$$

Thank you for your attention

Politecnico
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