Bilinear Bochner-Riesz Means Associated to the Sub-Laplacians on Métivier Groups and Grushin Operators

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Overview

- Linear Bochner-Riesz operator

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- Bilinear Bochner-Riesz operator

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- Bochner-Riesz means on Métivier groups
- Bochner-Riesz means for Grushin operators

Bochner-Riesz operator

- The Bochner-Riesz operator S^α of order $\alpha \geq 0$:

$$S^\alpha(f)(x) = \int_{\mathbb{R}^n} \left(1 - |\xi|^2\right)_+^\alpha \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi,$$

where $(r)_+ = \max\{r, 0\}$ for $r \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

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- Characterizing the optimal range of α such that S^α is bounded on $L^p(\mathbb{R}^n)$ is known as Bochner-Riesz conjecture.

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- For $0 < \alpha \leq \frac{n-1}{2}$:
  - Properties of Bessel function $\Longrightarrow$ S^α unbounded on $L^p(\mathbb{R}^n)$ if $\left|\frac{1}{p} - \frac{1}{2}\right| \geq \frac{2\alpha+1}{2n}$.

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- **Bochner-Riesz Conjecture:** For $1 \leq p \leq \infty$ and $p \neq 2$
  S^α bounded on $L^p(\mathbb{R}^n) \Longleftrightarrow \alpha > \alpha(p) = \max\left\{n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}$.

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  - S^α bounded on $L^p(\mathbb{R}^n)$. Using L^2 restriction on sphere (Stein-Tomas).

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- $n \geq 5$: True for $\max\{p, p'\} \geq 2 + 12/(4n - 3 - k)$ if $n \equiv k (\mod 3)$, $k = -1, 0, 1$ [Bourgain and Guth, 2011].

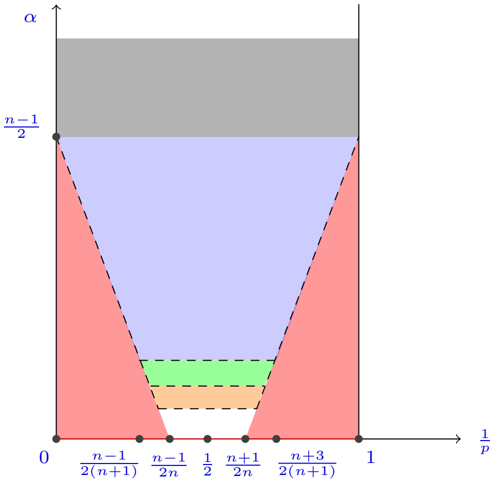

Figure: Red= Unbounded, Grey=Integrability, Blue=Stein-Tomas, Green= Sangyuk Lee, Orange= Bourgain and Guth for $n \geq 5$, White= open

Bilinear Bochner-Riesz

- The bilinear Bochner-Riesz operator B^α in $\mathbb{R}^n$ of order $\alpha \geq 0$:

$$B^\alpha(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(1 - |\xi|^2 - |\eta|^2\right)_+^\alpha \widehat{f}(\xi)\, \widehat{g}(\eta)\, e^{2\pi i x \cdot (\xi+\eta)}\, d\xi\, d\eta,$$

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- The bilinear Bochner-Riesz operator B^α in $\mathbb{R}^n$ of order $\alpha \geq 0$:

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- Find optimal range α:

$$B^\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

with $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$.

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- $\alpha > n - 1/2$:
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- $n = 1$: Solved $p_1, p_2, p \in [1, \infty]$. See [Grafakos and Li, 2006], [Bernicot et.al, 2015], [Jotsaroop and Shrivastava, 2022].

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- $n \geq 2$: B^α unbounded for $\alpha = 0$, if exactly one of p_1, p_2, p' is less than 2 [Diestel and Grafakos, 2007].

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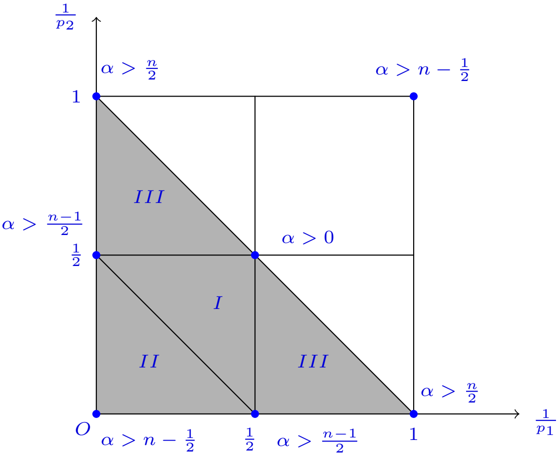
Banach triangle case



Banach triangle case

Theorem (Bernicot et. al, 2015)

Let $n \geq 2$ *and* $1 \leq p_1, p_2 \leq \infty$ *with* $1/p = 1/p_1 + 1/p_2$. *Then* B^α *is bounded from* $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$ *if* p_1, p_2, p *and* α *satisfy one of the following conditions:*

1. *(Region I)* $2 \leq p_1, p_2 < \infty$, $1 \leq p \leq 2$ *and* $\alpha > (n-1)(1-\frac{1}{p})$.
2. *(Region II)* $2 \leq p_1, p_2, p < \infty$ *and* $\alpha > \frac{n-1}{2} + n(\frac{1}{2} - \frac{1}{p})$.
3. *(Region III)* $2 \leq p_2 < \infty$, $1 \leq p_1, p < 2$ *and* $\alpha > n(\frac{1}{2} - \frac{1}{p_2}) - (1 - \frac{1}{p})$.
4. *(Region III)* $2 \leq p_1 < \infty$, $1 \leq p_2, p < 2$ *and* $\alpha > n(\frac{1}{2} - \frac{1}{p_1}) - (1 - \frac{1}{p})$.
5. $1 \leq p_1, p_2 \leq \infty$, $0 < p \leq \infty$ *and* $\alpha > n - \frac{1}{2}$.

- Improvement: Banach triangle case ($2 \leq p_1, p_2 \leq \infty$): [Jeong, Lee and Vargas, 2018]

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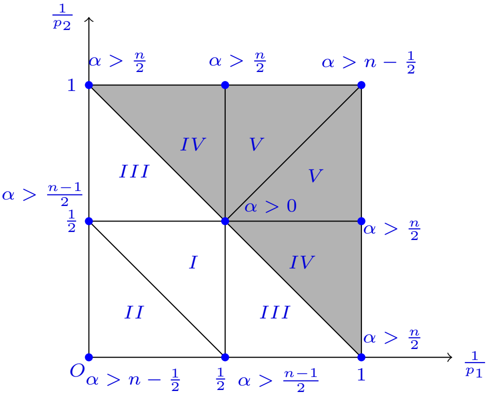
- Improvement: Non-Banach triangle ($p < 1$): [Liu and Wang, 2019]
    - B^α bounded $L^1 \times L^2 \to L^{2/3}$ for $\alpha > n/2$.



Non-Banach triangle case

Theorem (Liu and Wang, 2020)

Let $n \geq 2$ *and* $1 \leq p_1, p_2 \leq \infty$ *with* $1/p = 1/p_1 + 1/p_2$. *Then* B^α *is bounded from* $L^{p_1}(\mathbb{R}^n) \times L^{p_1}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$ *if* p_1, p_2, p *and* α *satisfy one of the following conditions:*

1. *(Region IV)* $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$, $0 < p < 1$ *and* $\alpha > n(\frac{1}{p_1} - \frac{1}{2})$.
2. *(Region IV)* $1 \leq p_2 \leq 2 \leq p_1 \leq \infty$, $0 < p < 1$ *and* $\alpha > n(\frac{1}{p_2} - \frac{1}{2})$.
3. *(Region V)* $1 \leq p_1 \leq p_2 \leq 2$ *and* $\alpha > n(\frac{1}{p} - 1) - (\frac{1}{p_2} - \frac{1}{2})$.
4. *(Region V)* $1 \leq p_2 \leq p_1 \leq 2$ *and* $\alpha > n(\frac{1}{p} - 1) - (\frac{1}{p_1} - \frac{1}{2})$.

Beyond the Euclidean setup

- Bochner-Riesz means:

$$S^\alpha f = (I - \Delta)_+^\alpha f,$$

where $\Delta = -\sum_{j=1}^n \partial_{x_j}^2$ the Euclidean Laplacian and $(s)_+ = \max\{s, 0\}$.

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- L: a non-negative self-adjoint operator. The Bochner-Riesz means associated L:

$$S^\alpha(L)f = (I - L)_+^\alpha f.$$

Bochner-Riesz means on Métivier groups

- G: two step stratified Lie group: connected, simply connected, two step nilpotent Lie group, Lie algebra endowed with a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}, \mathfrak{g}_2] = \{0\}$.

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- $d_1 = \dim\, \mathfrak{g}_1$, $d_2 = \dim\, \mathfrak{g}_2$.
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- $\langle \cdot, \cdot \rangle$ inner product on $\mathfrak{g}$ such that $X_1, \cdots, X_{d_1}, T_1, \ldots, T_{d_2}$ orthonormal basis of $\mathfrak{g}$.

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- G Heisenberg type group:

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- $|\cdot|$ norm on $\mathfrak{g}_2^*$ induced by $\langle \cdot, \cdot \rangle$.
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  - $\alpha > (d-1)/2$: Heisenberg-Reiter type groups [Martini, 2015], Lie groups of polynomial growth [Martini, 2012], Two-step stratified groups of small dimensions [Martini and Müller, 2014].

- $f \in L^1(G)$, $\mu \in \mathfrak{g}_2^*$. Partial Fourier transform of f along $\mathfrak{g}_2$:

$$f^\mu(x) = \int_{\mathfrak{g}_2} f(x,u) e^{-i\langle \mu, u\rangle}\, du, \quad x \in \mathfrak{g}_1.$$

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- Let $N \in \mathbb{N} \setminus \{0\}$, $\mathbf{b} \in (0,\infty)^N$, $\mathbf{r}, \mathbf{k} \in \mathbb{N}^N$.
    - $(\mathbf{b}, \mathbf{r})$-rescaled Laguerre functions

    $$\varphi_{\mathbf{k}}^{\mathbf{b},\mathbf{r}} = \varphi_{k_1}^{(b_1,r_1)} \otimes \cdots \otimes \varphi_{k_N}^{(b_N,r_N)},$$

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    $$\varphi_k^{(\lambda,m)}(z) = \lambda^m L_k^{m-1}(\tfrac{1}{2}\lambda|z|^2) e^{-\frac{1}{2}\lambda|z|^2}, \quad z \in \mathbb{R}^{2m},$$

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- μ-twisted convolution: For $\varphi, \psi \in \mathcal{S}(\mathfrak{g}_1)$

$$\phi \times_\mu \psi(x) = \int_{\mathfrak{g}_1} \phi(x')\psi(x-x') e^{\frac{i}{2}\omega_\mu(x,x')}\, dx', \quad x \in \mathfrak{g}_1.$$

- Bochner-Riesz means on Métivier groups:

$$S^\alpha(\mathcal{L})f(x,u) = \frac{1}{(2\pi)^{d_2}} \int_{\mathfrak{g}^*_{2,r}} \sum_{\mathbf{k}\in\mathbb{N}^N} \left(1-\lambda^\mu_{\mathbf{k}}\right)^\alpha_+ \left[f^\mu \times_\mu \varphi^{\mathbf{b}^\mu,\mathbf{r}}_{\mathbf{k}}(R^{-1}_\mu \cdot)\right](x)\, e^{i\langle \mu,u\rangle}\, d\mu,$$

- where $N \in \mathbb{N}\setminus\{0\}$, $\mathbf{r} = (r_1,\ldots,r_N) \in (\mathbb{N}\setminus\{0\})^N$, $\mathbf{b}^\mu \in (0,\infty)^N$
- $\mathfrak{g}^*_{2,r}$ is a Zariski open subset of $\mathfrak{g}^*_2$.
- $\lambda^\mu_{\mathbf{k}} = \sum_{n=1}^{N}(2k_n + r_n)b^\mu_n$,
- the function $\mu \to b^\mu_n$ are homogeneous of degree 1 and continuous on $\mathfrak{g}^*_2$, real analytic on $\mathfrak{g}^*_{2,r}$, and satisfy $b^\mu_n > 0$ for all $\mu \in \mathfrak{g}^*_2\setminus\{0\}$,
- $\mu \to R_\mu \in O(d_1)$ is a Borel measurable function on $\mathfrak{g}^*_{2,r}$ which is homogeneous of degree 0.

- $\alpha > (d-1)/2$: $S^\alpha(\mathcal{L})$ bounded on $L^p(G)$ for $1 < p < \infty$ [Martini, 2012].

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- $0 < \alpha \leq (d-1)/2$
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    For $1 \leq p \leq p_{d_1,d_2}$ whenever $\alpha > d(1/p - 1/2) - 1/2$, then $S^\alpha(\mathcal{L})$ bounded on $L^p(G)$ [Niedorf, 2025].

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- In fact it follows from [Martini et. al, 2022] that the above results are sharp.

Bilinear Bochner-Riesz means for $\mathcal{L}$

- $f, g \in \mathcal{S}(G)$, bilinear Bochner-Riesz operator:

$$\mathcal{B}^{\alpha}(f,g)(x,u) = \frac{1}{(2\pi)^{2d_2}} \int_{\mathfrak{g}_{2,r}^*} \int_{\mathfrak{g}_{2,r}^*} e^{i\langle \mu_1+\mu_2, u\rangle} \sum_{\mathbf{k}_1,\mathbf{k}_2 \in \mathbb{N}^N} \left(1 - \lambda_{\mathbf{k}_1}^{\mu_1} - \lambda_{\mathbf{k}_2}^{\mu_2}\right)_+^{\alpha}$$

$$\left[f^{\mu_1} \times_{\mu_1} \varphi_{\mathbf{k}_1}^{\mathbf{b}^{\mu_1},\mathbf{r}_1}(R_{\mu_1}^{-1}\cdot)\right](x) \left[g^{\mu_2} \times_{\mu_2} \varphi_{\mathbf{k}_2}^{\mathbf{b}^{\mu_2},\mathbf{r}_2}(R_{\mu_2}^{-1}\cdot)\right](x)\, d\mu_1\, d\mu_2.$$

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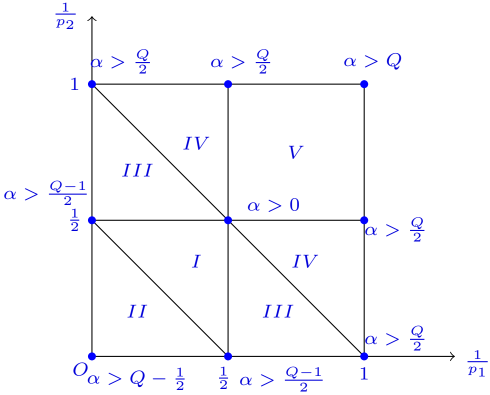
- Find $\alpha(p_1, p_2)$ in terms of the topological dimension $d = d_1 + d_2$ such that whenever $\alpha > \alpha(p_1, p_2)$,

$$\|\mathcal{B}^{\alpha}(f,g)\|_{L^p(G)} \leq C\|f\|_{L^{p_1}(G)}\|g\|_{L^{p_2}(G)},$$

with $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$.

- Bilinear Bochner-Riesz means outside the Euclidean spaces.
  - Sub-Laplacians on Heisenberg group, see [Liu and Wang, 2019].
  - Heisenberg type groups, see [Wang and Wang, 2024].



Theorem (Wang and Wang, 2024)

Assume $1 \le p_1, p_2 \le \infty$ *with* $1/p = 1/p_1 + 1/p_2$. *Then* $\mathcal{B}^\alpha$ *is bounded from* $L^{p_1}(\mathbb{H}) \times L^{p_1}(\mathbb{H})$ *to* $L^p(\mathbb{H})$ *if* p_1, p_2, p *and* $\alpha > \alpha(p_1, p_2)$ *satisfy one of the following conditions:*

1. *(Region I)* $2 \le p_1, p_2 < \infty$, $1 \le p \le 2$, $\alpha(p_1, p_2) = (Q-1)(1-\frac{1}{p})$.
2. *(Region II)* $2 \le p_1, p_2, p < \infty$, $\alpha(p_1, p_2) = \frac{Q-1}{2} + Q(\frac{1}{2} - \frac{1}{p})$.
3. *(Region III)* $2 \le p_2 < \infty$, $1 \le p_1, p < 2$, $\alpha(p_1, p_2) = Q(\frac{1}{2} - \frac{1}{p_2}) - (1 - \frac{1}{p})$.
4. *(Region III)* $2 \le p_1 < \infty$, $1 \le p_2, p < 2$, $\alpha(p_1, p_2) = Q(\frac{1}{2} - \frac{1}{p_2}) - (1 - \frac{1}{p})$.
5. *(Region IV)* $1 \le p_1 \le 2 \le p_2 \le \infty$, $0 < p < 1$, $\alpha(p_1, p_2) = Q(\frac{1}{p_1} - \frac{1}{2})$.
6. *(Region IV)* $1 \le p_2 \le 2 \le p_1 \le \infty$, $0 < p < 1$, $\alpha(p_1, p_2) = Q(\frac{1}{p_1} - \frac{1}{2})$.
7. *(Region V)* $1 \le p_1, p_2 \le 2$, $\alpha(p_1, p_2) = Q(\frac{1}{p} - 1)$.

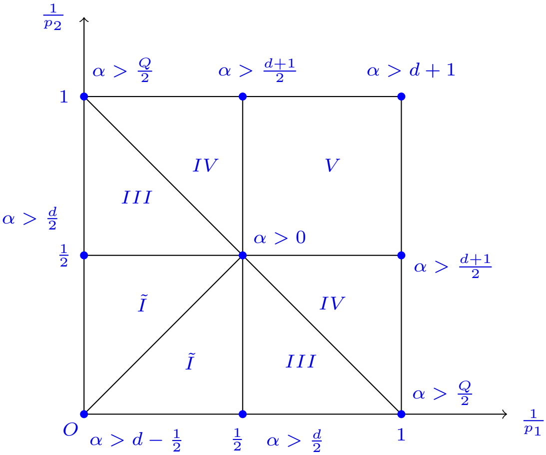
Boundedness of $\mathcal{B}^\alpha$

- G : Métivier groups

We have obtained the following result.

Theorem

- $\mathcal{B}^\alpha$ *is bounded from* $L^2(G) \times L^2(G) \to L^1(G)$ *if* $\alpha > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^2(G) \times L^\infty(G) \to L^2(G)$ *if* $\alpha > d/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^\infty(G) \times L^\infty(G) \to L^\infty(G)$ *if* $\alpha > d - 1/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^1(G) \to L^{1/2}(G)$ *if* $\alpha > d + 1$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^2(G) \to L^{2/3}(G)$ *if* $\alpha > (d+1)/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^\infty(G) \to L^1(G)$ *if* $\alpha > Q/2$.



$\frac{1}{p_2}$
$\alpha > \frac{Q}{2}$
$\alpha > \frac{d+1}{2}$
$\alpha > d+1$
1
IV
V
III
$\alpha > \frac{d}{2}$
$\alpha > 0$
$\frac{1}{2}$
$\alpha > \frac{d+1}{2}$
$\tilde{I}$
IV
$\tilde{I}$
III
$\alpha > \frac{Q}{2}$
$\frac{1}{p_1}$
O
$\alpha > d - \frac{1}{2}$
$\frac{1}{2}$
$\alpha > \frac{d}{2}$
1

Theorem

Let $1 \leq p_1, p_2 \leq \infty$ *with* $1/p = 1/p_1 + 1/p_2$. *Then* $\mathcal{B}^\alpha$ *is bounded from* $L^{p_1}(G) \times L^{p_1}(G)$ *to* $L^p(G)$ *if* p_1, p_2, p *and* $\alpha > \alpha(p_1, p_2)$ *satisfy one of the following conditions:*

1. *(Region $\tilde{I}$)* $2 \leq p_2 \leq p_1 \leq \infty$, $1 \leq p \leq \infty$,
   $\alpha(p_1, p_2) = (d - \frac{1}{2})(1 - \frac{2}{p_2}) + d(\frac{1}{p_2} - \frac{1}{p_1})$ *and also with* p_1, p_2 *interchanging.*
2. *(Region III)* $2 \leq p_2 \leq \infty$, $1 \leq p_1, p \leq 2$,
   $\alpha(p_1, p_2) = Q(\frac{1}{p_1} - \frac{1}{2}) + (d - 1)(1 - \frac{1}{p})$ *and also with* p_1, p_2 *interchanging.*
3. *(Region IV)* $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$, $0 < p \leq 1$,
   $\alpha(p_1, p_2) = (d + 1)(\frac{1}{p} - 1) + Q(\frac{1}{2} - \frac{1}{p_2})$ *and also with* p_1, p_2 *interchanging.*
4. *(Region V)* $1 \leq p_1, p_2 \leq 2$ *and* $\alpha(p_1, p_2) = (d + 1)(\frac{1}{p} - 1)$.

Improvement

Theorem

- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^\infty(G) \to L^1(G)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 g(x, \cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^\infty(G) \times L^1(G) \to L^1(G)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 f(x, \cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^2(G) \to L^{2/3}(G)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 g(x, \cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
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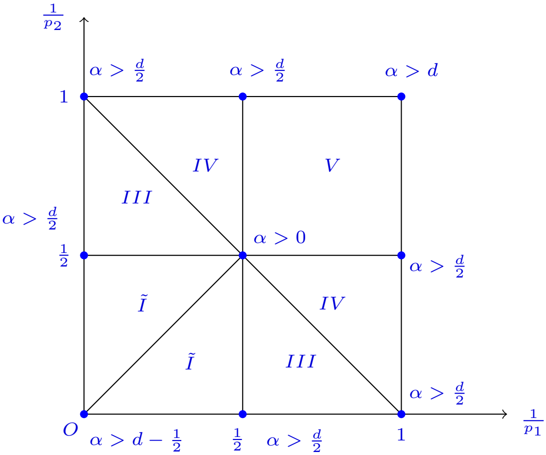
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(G) \times L^1(G) \to L^{1/2}(G)$ *if* $\alpha > d$ *and* $\operatorname{supp} \mathcal{F}_2 f(x, \cdot) \subseteq \{|\mu_1| \geq \delta_0\}$ *and* $\operatorname{supp} \mathcal{F}_2 g(x, \cdot) \subseteq \{|\mu_2| \geq \delta_1\}$ *for some* $\delta_0, \delta_1 > 0$.



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1

Bochner-Riesz means for Grushin operators

Grushin Operator

- Grushin operator $\mathcal{L}$ on $\mathbb{R}^d$:

$$\begin{aligned}\mathcal{L} &= -\sum_{j=1}^{d_1} \partial^2_{x'_j} - \left(\sum_{j=1}^{d_1} |x'_j|^2\right) \sum_{k=1}^{d_2} \partial^2_{x''_k} \\ &= -\Delta_{x'} - |x'|^2 \Delta_{x''},\end{aligned}$$

where $x = (x', x'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $\Delta_{x'}$, $\Delta_{x''}$ Laplacian on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ respectively and $|x'|$ the Euclidean norm of x'.

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&= -\Delta_{x'} - |x'|^2 \Delta_{x''},
\end{aligned}
$$

  where $x = (x', x'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $\Delta_{x'}$, $\Delta_{x''}$ Laplacian on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ respectively and $|x'|$ the Euclidean norm of x'.

- $\mathcal{L}$ is positive, essentially self-adjoint on $L^2(\mathbb{R}^d)$. However $\mathcal{L}$ is not elliptic on the plane $x' = 0$.

- Carnot-Carathéodory distance for $x, y \in \mathbb{R}^d$:

$$\varrho(x,y) \sim |x'-y'| + \begin{cases} \frac{|x''-y''|}{|x'|+|y'|} & \text{if } |x''-y''|^{1/2} \leq |x'|+|y'| \\ |x''-y''|^{1/2} & \text{if } |x''-y''|^{1/2} \geq |x'|+|y'|. \end{cases}$$

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- If $B(x,r) := \{y \in \mathbb{R}^d : \varrho(x,y) < r\}$, then

$$|B(x,r)| \sim r^{d_1+d_2} \max\{r, |x'|\}^{d_2}.$$

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- We have $|B(x,\eta r)| \le C(1+\eta)^Q |B(x,r)|$ for $\eta \ge 0$ and $Q = d_1 + 2d_2$.

- Carnot-Carathéodory distance for $x, y \in \mathbb{R}^d$:

$$\varrho(x,y) \sim |x'-y'| + \begin{cases} \frac{|x''-y''|}{|x'|+|y'|} & \text{if } |x''-y''|^{1/2} \le |x'|+|y'| \\ |x''-y''|^{1/2} & \text{if } |x''-y''|^{1/2} \ge |x'|+|y'|. \end{cases}$$

- If $B(x,r) := \{y \in \mathbb{R}^d : \varrho(x,y) < r\}$, then

$$|B(x,r)| \sim r^{d_1+d_2} \max\{r, |x'|\}^{d_2}.$$

- We have $|B(x,\eta r)| \le C(1+\eta)^Q |B(x,r)|$ for $\eta \ge 0$ and $Q = d_1 + 2d_2$.

- $Q := d_1 + 2d_2$=homogeneous dimension of $\mathbb{R}^d$.

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- $Q := d_1 + 2d_2$=homogeneous dimension of $\mathbb{R}^d$.

- $d := d_1 + d_2$=topological dimension of $\mathbb{R}^d$.

The Bochner-Riesz operator $S^\alpha(\mathcal{L})$ defined by,

$$S^\alpha(\mathcal{L})f(x) = \frac{1}{(2\pi)^{d_2}} \int_{\mathbb{R}^{d_2}} \sum_{k=0}^{\infty} (1-(2k+d_1)|\lambda|)_+^\alpha \, P_k^\lambda f^\lambda(x') e^{i\lambda\cdot x''} \; d\lambda.$$

where

$$f^\lambda(x') = \int_{\mathbb{R}^{d_2}} f(x,x'') e^{-i\lambda\cdot x''} \; dx'', \quad P_k^\lambda g(x') = \sum_{|\mu|=k} \langle g, \Phi_\mu^\lambda \rangle \Phi_\mu^\lambda(x'),$$

and for $\lambda \neq 0$, $\Phi_\mu^\lambda(x') := |\lambda|^{d_1/4} \Phi_\mu(|\lambda|^{1/2} x')$ is called the scaled Hermite functions on $\mathbb{R}^{d_1}$.

- Boundedness of $S^{\alpha}(\mathcal{L})$ for $1 < p < \infty$.
  - If $\alpha > (\max\{d_1 + d_2, 2d_2\} - 1)/2$ by [Martini and Sikora, 2012].

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For $0 < \alpha \leq (d - 1)/2$.

- Boundedness of $S^\alpha(\mathcal{L})$ for $1 \leq p \leq \min\{2d_1/(d_1 + 2), (2d_2 + 2)/(d_2 + 3)\}$.
  - If $\alpha > \max\{d_1 + d_2, 2d_2\}(1/p - 1/2) - 1/2$ by [Chen and Ouhabaz, 2016].

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  - If $\alpha > d(1/p - 1/2) - 1/2$ by [Niedorf, 2022].
- In fact it follows from [Martini et. al, 2022] that the above results with d are sharp.

Bilinear Bochner-Riesz means

The bilinear Bochner-Riesz means associated with $\mathcal{L}$ is defined by

$$\mathcal{B}^\alpha(f,g)(x) = \frac{1}{(2\pi)^{2d_2}} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_2}} e^{i(\lambda_1+\lambda_2)\cdot x''} \sum_{k_1,k_2=0}^{\infty} (1 - [k_1]|\lambda_1| - [k_2]|\lambda_2|)_+^\alpha \\ P_{k_1}^{\lambda_1} f^{\lambda_1}(x') P_{k_2}^{\lambda_2} g^{\lambda_2}(x') \; d\lambda_1 \; d\lambda_2,$$

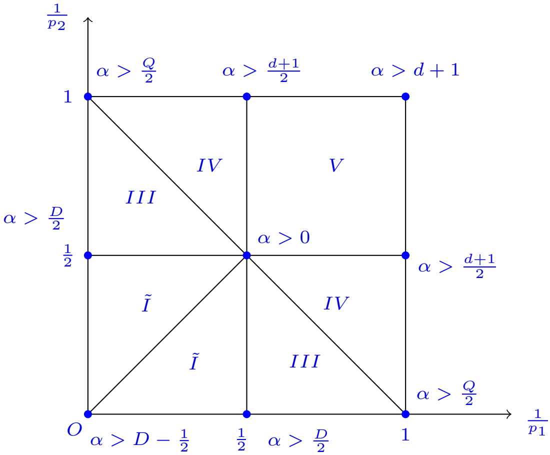
for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Boundedness of $\mathcal{B}^\alpha$

We have obtained the following result.

Theorem

- $\mathcal{B}^\alpha$ *is bounded from* $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ *if* $\alpha > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^2(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *if* $\alpha > d/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^\infty(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ *if* $\alpha > d - 1/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^{1/2}(\mathbb{R}^d)$ *if* $\alpha > d + 1$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^{2/3}(\mathbb{R}^d)$ *if* $\alpha > (d + 1)/2$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ *if* $\alpha > Q/2$.



$\frac{1}{p_2}$
$\alpha > \frac{Q}{2}$
$\alpha > \frac{d+1}{2}$
$\alpha > d+1$
1
IV
V
III
$\alpha > \frac{D}{2}$
$\alpha > 0$
$\frac{1}{2}$
$\alpha > \frac{d+1}{2}$
$\tilde{I}$
IV
$\tilde{I}$
III
$\alpha > \frac{Q}{2}$
O
$\alpha > D - \frac{1}{2}$
$\frac{1}{2}$
$\alpha > \frac{D}{2}$
1
$\frac{1}{p_1}$

Theorem

Let $1 \leq p_1, p_2 \leq \infty$ *and* $p \geq 1$ *with* $1/p = 1/p_1 + 1/p_2$. *Then* $\mathcal{B}^\alpha$ *is bounded from* $L^{p_1}(\mathbb{R}^d) \times L^{p_1}(\mathbb{R}^d)$ *to* $L^p(\mathbb{R}^d)$ *if* p_1, p_2, p *and* $\alpha > \alpha(p_1, p_2)$ *satisfy one of the following conditions:*

- *(Region $\tilde{I}$)* $2 \leq p_2 \leq p_1 \leq \infty$, $1 \leq p \leq \infty$ *and* $\alpha(p_1, p_2) = (D - \frac{1}{2})(1 - \frac{2}{p_2}) + D(\frac{1}{p_2} - \frac{1}{p_1})$ *and interchanging* p_1, p_2.
- *(Region III)* $2 \leq p_2 \leq \infty$, $1 \leq p_1, p \leq 2$ *and* $\alpha(p_1, p_2) = Q(\frac{1}{p_1} - \frac{1}{2}) + (D - 1)(1 - \frac{1}{p})$ *and interchanging* p_1, p_2.
- *(Region IV)* $1 \leq p_1 \leq 2 \leq p_2 \leq \infty$, $0 < p \leq 1$ *and* $\alpha(p_1, p_2) = (d + 1)(\frac{1}{p} - 1) + Q(\frac{1}{2} - \frac{1}{p_2})$ *and interchanging* p_1, p_2.
- *(Region V)* $1 \leq p_1, p_2 \leq 2$ *and* $\alpha(p_1, p_2) = (d + 1)(\frac{1}{p} - 1)$.

For the case $d_1 \geq d_2$

- $\mathcal{B}^\alpha$ is bounded from $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^{1/2}(\mathbb{R}^d)$ if $\alpha > d$.

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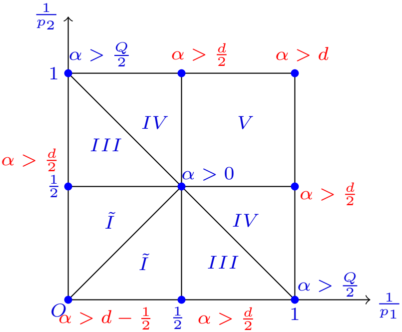
- $\mathcal{B}^\alpha$ is bounded from $L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^{2/3}(\mathbb{R}^d)$ if $\alpha > d/2$.



Improvement

Theorem

- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 g(x,\cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^\infty(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 f(x,\cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^{2/3}(\mathbb{R}^d)$ *if* $\alpha > d/2$ *and* $\operatorname{supp} \mathcal{F}_2 g(x,\cdot) \subseteq \{|\mu_2| \geq \delta_0\}$ *for some* $\delta_0 > 0$.
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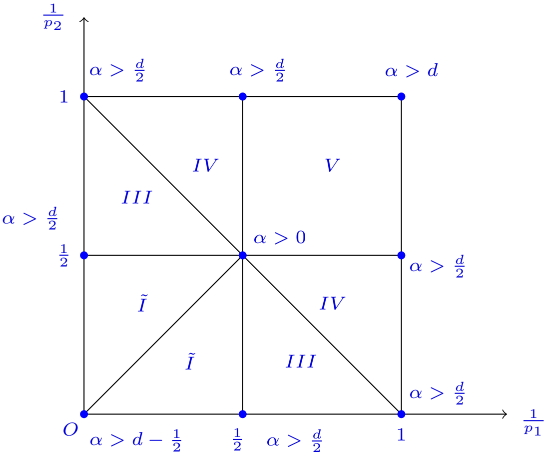
- $\mathcal{B}^\alpha$ *is bounded from* $L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d) \to L^{1/2}(\mathbb{R}^d)$ *if* $\alpha > d$ *and* $\operatorname{supp} \mathcal{F}_2 f(x,\cdot) \subseteq \{|\mu_1| \geq \delta_0\}$ *and* $\operatorname{supp} \mathcal{F}_2 g(x,\cdot) \subseteq \{|\mu_2| \geq \delta_1\}$ *for some* $\delta_0, \delta_1 > 0$.



$\frac{1}{p_2}$
$\alpha > \frac{d}{2}$
$\alpha > \frac{d}{2}$
$\alpha > d$
1
IV
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$\alpha > 0$
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$\alpha > \frac{d}{2}$
$\tilde{I}$
IV
$\tilde{I}$
III
$\alpha > \frac{d}{2}$
$\frac{1}{p_1}$
O
$\alpha > d - \frac{1}{2}$
$\frac{1}{2}$
$\alpha > \frac{d}{2}$
1

Thank you!