Resonances for pseudo-Riemannian hyperbolic spaces

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Outline

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Studied intensely by Borthwick, Bunke, Delarue, Guillarmou, Guillopé, Hilgert, Mazzeo, Melrose, Olbrich, Pasquale, Perry, Przebinda, Roby, Sjöstrand, Strohmaier, Vasy, Weich, Zworski, ...

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Credo

Resonances are a way of associating to a non-compact Riemannian manifold a discrete set of spectral invariants similar to the set of eigenvalues of the Laplacian on a compact Riemannian manifold.

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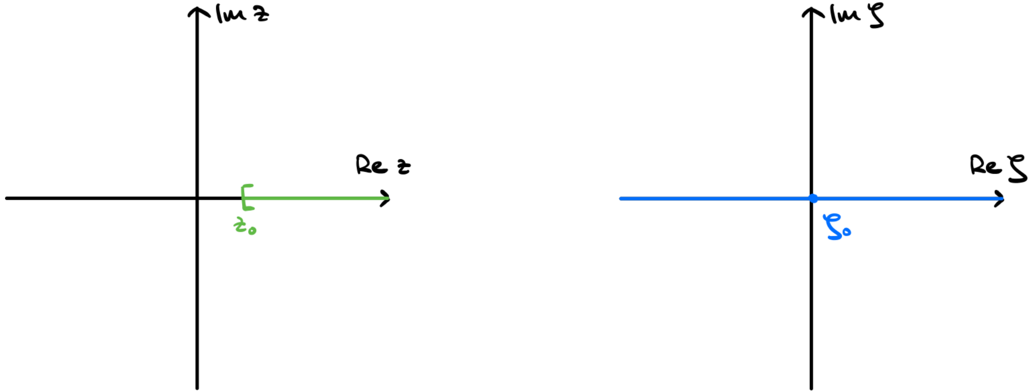

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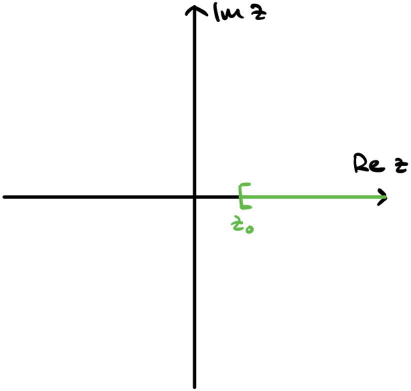

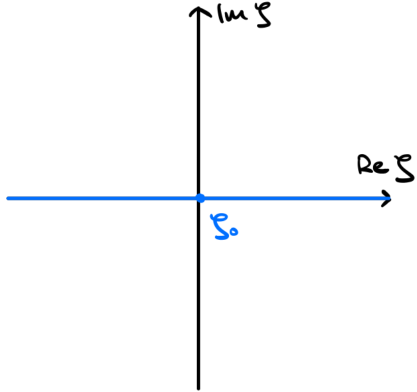

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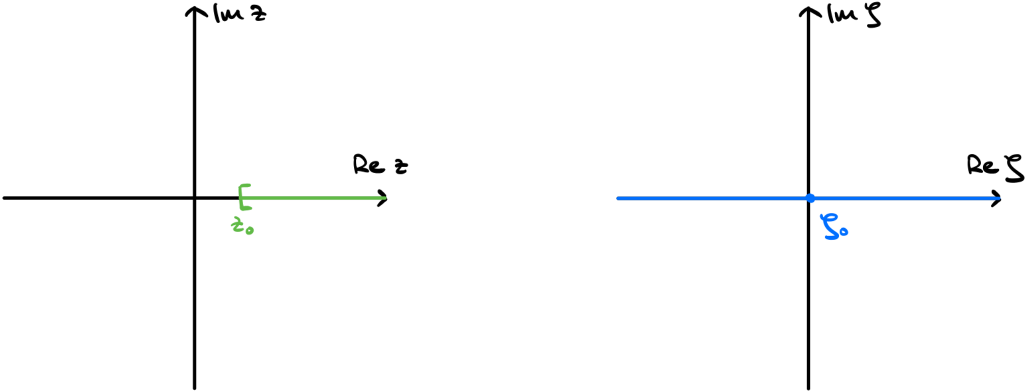

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$\rightsquigarrow$ *Goal:* Extend $\widetilde{R}(\zeta)$ from $\mathbb{C}_+ = \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$ across $\mathbb{R}$ to $\mathbb{C}$

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$\rightsquigarrow$ $\widetilde{R}(\zeta)$ is meromorphic in $\zeta \in \mathbb{C}$ with a simple pole at $\zeta = 0$ and $\operatorname{Res}_{\zeta=0} \widetilde{R}(\zeta) f(x) = \pi i \widehat{f}(0)$.

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Idea: Study the same problem for some pseudo-Riemannian symmetric spaces of rank one

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Geometric definition

The quadratic form $Q(z) = -|z_1|^2 - \cdots - |z_p|^2 + |z_{p+1}|^2 + \cdots + |z_{p+q}|^2$ on $\mathbb{F}^{p+q}$ induces a pseudo-Riemannian metric of signature $(dq, d(p-1))$ on

$$X = \{z \in \mathbb{F}^{p+q} : Q(z) = -1\}/U(1; \mathbb{F}),$$

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Group-theoretic definition

$G = \mathrm{U}(p, q; \mathbb{F}) = \{g \in \mathrm{GL}(p+q, \mathbb{F}) : Q(gz) = Q(z) \, \forall \, z \in \mathbb{F}^{p+q}\} = \mathrm{O}(p, q), \mathrm{U}(p, q), \mathrm{Sp}(p, q)$ acts transitively on X from the left and leaves the metric invariant, so we can identify

$$X \simeq G/H = \mathrm{U}(p, q; \mathbb{F})/(\mathrm{U}(1; \mathbb{F}) \times \mathrm{U}(p-1, q; \mathbb{F})).$$

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- X is Riemannian if and only if $p = 1$:

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Exceptional case

One more space along the same lines: $X = \mathrm{U}(2,1;\mathbb{O})/\mathrm{U}(1,1;\mathbb{O}) = F_{4(-20)}/\mathrm{Spin}_0(1,8)$

Outline

The Laplace–Beltrami operator

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$\rightsquigarrow$ consider the resolvent $\widetilde{R}(\zeta) = (\Box - \rho^2 - \zeta^2)^{-1}$

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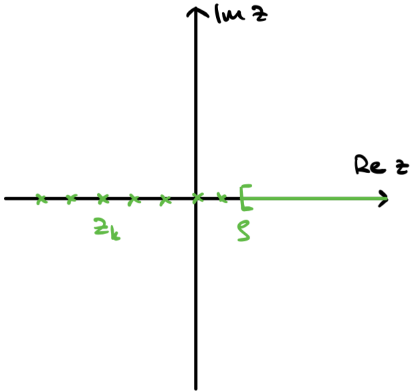

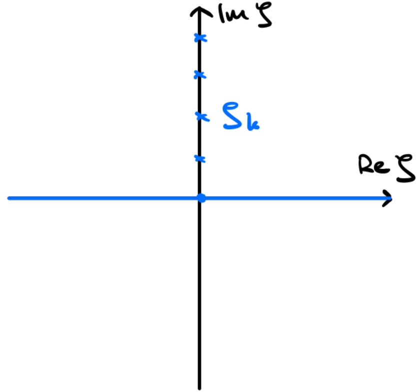

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$$z = \rho^2 + \zeta^2$$

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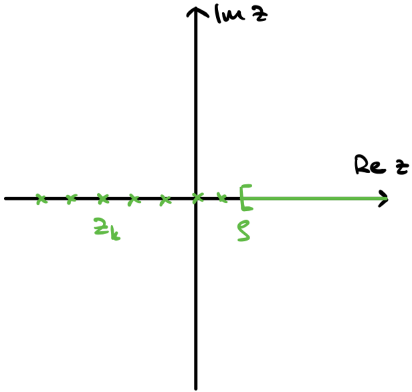

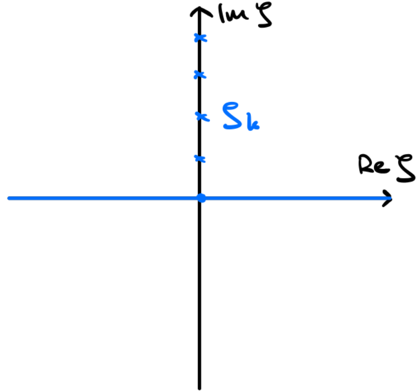

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Goal: Extend $\widetilde{R}(\zeta)$ from $\mathbb{C}_+ \setminus \{\zeta_k = i\sqrt{\rho^2 - z_k} : k \in \mathbb{N}\}$ across $\mathbb{R}$ to all of $\mathbb{C}$

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For $f \in C_c^\infty(X)$:

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Note: $f \mapsto f * \varphi_s$ can be written as the composition of a *Fourier transform* and a *Poisson transform*, both related to functions on the "boundary" of X:

$$\Xi = \{z \in \mathbb{F}^{p+q} \setminus \{0\} : Q(z) = 0\}/\mathrm{U}(1;\mathbb{F}).$$

Outline

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3. Study possible cancellation between the residues and the discrete part

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   $\rightsquigarrow$ extend to $f \in C_c^\infty(X)$ (not necessarily K-finite)

1. Justify contour shift: Paley–Wiener Theorem

$$\widetilde{R}(\zeta)f(x) = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{(f * \varphi_{i\nu})(x)}{\nu^2 - \zeta^2} \frac{d\nu}{c(i\nu)c(-i\nu)} + \sum_k c_k \cdot \frac{(f * \psi_k)(x)}{\zeta^2 - \zeta_k^2}.$$

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How does $\frac{(f * \varphi_{i\nu})(x)}{c(i\nu)c(-i\nu)}$ behave when $\operatorname{Re}\nu \to \pm\infty$ and $|\operatorname{Im}\nu| \leq N$?

Proposition (F.–Spilioti '23)

Let $f \in C_c^\infty(X)$. For $\Omega \subseteq X$ compact, $N > 0$ and $M \in \mathbb{N}$ there exists $C > 0$ such that

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Note: Only the *easy* part of a full Paley–Wiener type theorem.

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  $\rightsquigarrow$ the residue of the product $(c(i\nu)c(-i\nu))^{-1}\cdot(f*\varphi_{i\nu})(x)$ can be complicated
  (in the residue formula only the term of degree -1 in the Laurent expansion contributes)

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- sometimes double poles (e.g. for $\mathbb{F}=\mathbb{R}$ with p even and q odd)

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2. The resonances are:
   - In the upper half plane: ζ_k, $k \in \mathbb{N}$.
   - On the real line: 0 iff $\mathbb{F} = \mathbb{R}$ and $p - q \equiv 2(4)$ or $\mathbb{F} = \mathbb{C}$ and $p - q \equiv 0(2)$.
   - In the lower half plane: $-is$, $s \in \rho + 2\mathbb{N}$ if either $\mathbb{F} = \mathbb{R}$ with p, q even or $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$
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3. For each resonance ζ_0, the image of the residue operator $\operatorname{Res}_{\zeta=\zeta_0} \widetilde{R}(\zeta) : C_c^\infty(X) \to \mathcal{D}'(X)$ is an irreducible representation of $G = \mathrm{U}(p, q; \mathbb{F})$:
   - For $\zeta_0 = \zeta_k$, $k \in \mathbb{N}$, it is the discrete series representation $\{u \in L^2(X) : \Box u = z_k u\}$
   - For $\zeta_0 = 0$ it is a limit of discrete series.
   - If either $\mathbb{F} = \mathbb{R}$ with p, q even or $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, then for $\zeta_0 = -is$, $s \in \rho + 2\mathbb{N}$, the residue representation is finite-dimensional of highest weight ...
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