Clark Measures on Symmetric Domains

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Riesz–Herglotz representation theorem

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. If f is a positive harmonic function on $\mathbb{D}$, then there is a unique positive Radon measure μ on $\mathbb{T}$ such that

$$f(z) = (\mathcal{P}\mu)(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\alpha - z|^2}\, \mathrm{d}\mu(\alpha)$$

for every $z \in \mathbb{D}$. In addition, μ is the vague limit of $f_\rho \cdot \beta_{\mathbb{T}}$ for $\rho \to 1^-$, where $f_\rho(\alpha) = f(\rho\alpha)$ for every $\alpha \in \mathbb{T}$ and $\beta_{\mathbb{T}}$ is the normalized Haar measure on $\mathbb{T}$.

Equivalently, if g is a holomorphic function on $\mathbb{D}$ with a positive real part, then there is a unique positive Radon measure ν on $\mathbb{T}$ such that

$$g(z) = (\mathcal{H}\nu)(z) + i\,\mathrm{Im}\, g(0) = \int_{\mathbb{T}} \frac{\alpha+z}{\alpha-z}\, \mathrm{d}\nu(\alpha) + i\,\mathrm{Im}\, g(0).$$

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Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. For every $\alpha \in \mathbb{T}$, there is a unique positive pluriharmonic measure μ_α on $\mathbb{T}$ such that

$$\mathcal{P}\mu_\alpha = \operatorname{Re}\frac{\alpha+\varphi}{\alpha-\varphi} = \frac{1-|\varphi|^2}{|\alpha-\varphi|^2}.$$

Denote by μ_α^s its singular part with respect to $\beta_\mathbb{T}$.

- $\|\mu_\alpha\|_{\mathcal{M}^1(\mathbb{T})} = \frac{1-|\varphi(0)|^2}{|\alpha-\varphi(0)|^2}$;
- $\mu_\alpha = \frac{1-|\varphi_1|^2}{|\alpha-\varphi_1|^2}\cdot\beta_\mathbb{T} + \mu_\alpha^s$, where φ_1 denotes the a.e. limit of $\varphi_\rho = \varphi(\rho\,\cdot)$, $\rho \to 1^-$;
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- μ_α^s and $\mu_{\alpha'}^s$ are singular for $\alpha \neq \alpha'$;

- ('Aleksandrov's disintegration theorem')

$$\int_{\mathbb{T}} \mu_\alpha \, \mathrm{d}\beta_{\mathbb{T}}(\alpha) = \beta_{\mathbb{T}};$$

- if φ is inner (i.e., $|\varphi_1(\alpha)| = 1$ for a.e. $\alpha \in \mathbb{T}$), then (μ_α) is a disintegration of $\beta_{\mathbb{T}}$ relative to $\beta_{\mathbb{T}}$;
- if φ is inner and $\varphi(0) = 0$, then $(\varphi_1)_*(\beta_{\mathbb{T}}) = \beta_{\mathbb{T}}$;
- if $\varphi(0) = 0$, then

$$\int_{\mathbb{T}} \overline{\zeta}^k \, \mathrm{d}\mu_\alpha(\zeta) = \sum_{h=1}^{k} \overline{\alpha}^h \int_{\mathbb{T}} \varphi^h(\zeta) \overline{\zeta}^k \, \mathrm{d}\beta_{\mathbb{T}}(\zeta)$$

for every $k \in \mathbb{N}$.

The mapping $\mathcal{C}_\alpha \colon L^2(\mu_\alpha) \to H^2(\mathbb{D})$,

$$\mathcal{C}_\alpha(f)(z) = (1 - \overline{\alpha}\varphi(z)) \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \overline{\zeta} z} \, \mathrm{d}\mu_\alpha(\zeta),$$

is contractive. If $H^2(\mu_\alpha)$ denotes the closure of the space of *holomorphic* polynomials in $L^2(\mu_\alpha)$, then $\ker \mathcal{C}_\alpha = H^2(\mu_\alpha)^\perp$.

When φ is inner and $\varphi(0) = 0$, then $\mathcal{C}_\alpha$ is an isometry of $L^2(\mu_\alpha)$ onto the 'model space' $(\varphi H^2(\mathbb{D}))^\perp$ with inverse $f \mapsto f_1$ (the pointwise a.e. limit of $f(\rho\,\cdot\,)$ for $\rho \to 1^-$). In addition,

$$\mathcal{C}_\alpha S = U_\alpha \mathcal{C}_\alpha$$

where S denotes multiplication by z (the 'shift') and U_α is a rank-one perturbation of the 'compressed shift' on $(\varphi H^2(\mathbb{D}))^\perp$. This provides a connection with the theory of contractions.

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When $\varphi\colon \mathbb{D} \to \mathbb{D}$ is a rational inner function, then it is a finite Blashke product, that is, of the form

$$\varphi(z) = \alpha' z^k \prod_{j=1}^{h} \frac{z - \lambda_j}{1 - \overline{\lambda_j} z},$$

with $\alpha' \in \mathbb{T}$, $\lambda_1, \dots, \lambda_h \in \mathbb{D}$. Then,

$$\mu_\alpha = \sum_{\varphi(\alpha'')=\alpha} \frac{1}{|\varphi'(\alpha'')|} \delta_{\alpha''}.$$

In general, $H^2(\mu_\alpha) = L^2(\mu_\alpha)$ if and only if φ is extremal in the (closed) unit ball of $H^\infty(\mathbb{D})$. *This does not depend on* $\alpha \in \mathbb{T}$.

Let D be a convex circular bounded symmetric domain, i.e., for every $z \in D$ there is an involutive biholomorphism of D with z as its unique fixed point.
Then, the group K of its *linear* automorphisms acts transitively on the set of extremal points (the 'Šilov boundary') $\mathrm{b}D$ of D. Let $\beta_{\mathrm{b}D}$ be the normalized K-invariant measure on $\mathrm{b}D$.

Examples

- $D = U$, the unit ball in $\mathbb{C}^n$, has $K = U(n)$ and $\mathrm{b}D = \partial U$;
- $D = \mathbb{D}^n$, the unit polydisc in $\mathbb{C}^n$, has $K = \mathfrak{S}_n \times \mathbb{T}^n$, with action $(\sigma, (\alpha_j)) \cdot (z_j) = (\alpha_{\sigma(j)} z_{\sigma(j)})$, and $\mathrm{b}D = \mathbb{T}^n (\subsetneq \partial D)$;
- $D = \{A \in M_n(\mathbb{C}) \colon \|A\| < 1\}$ has $K = \mathbb{T} \times SU(n)^2$, with action $(\alpha, B, C) \cdot A = \alpha B A C^{-1}$, and $\mathrm{b}D = U(n)$.

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One may then define the Hardy space $H^2(D)$ as

$$\left\{ f \in \mathrm{Hol}(D) \colon \sup_{\rho \in (0,1)} \int_{\mathrm{b}D} |f(\rho\zeta)|^2 \, \mathrm{d}\beta_{\mathrm{b}D}(\zeta) < \infty \right\}.$$

Then, $H^2(D)$ is a Hilbert space and $f \mapsto f_1 = \lim_{\rho \to 1^-} f_\rho$ (limit in $L^2(\beta_{\mathrm{b}D})$, $f_\rho(\zeta) = f(\rho\zeta)$) is an isometry of $H^2(D)$ into $L^2(\beta_{\mathrm{b}D})$. In addition, point evaluations are continuous on $H^2(D)$, so that for every $z \in D$ there is a unique $\mathcal{C}_z \in H^2(D)$ such that

$$f(z) = \langle f | \mathcal{C}_z \rangle_{H^2(D)}$$

for every $f \in H^2(D)$.

- if $D = U$, the unit ball in $\mathbb{C}^n$, then $\mathcal{C}_z(z') = (1 - \langle z | z' \rangle^n)^{-1}$;
- if $D = \mathbb{D}^n$, then $\mathcal{C}_{z'}(z) = \prod_j (1 - z_j \overline{z'_j})^{-1}$;
- if D is the unit ball of $M_n(\mathbb{C})$, then $\mathcal{C}_{z'}(z) = (\det(I - z z'^*))^{-1}$.

The function $\mathcal{C}(z,z') = \mathcal{C}_{z'}(z)$ is the reproducing kernel of $H^2(D)$ (the 'Cauchy–Szegő' kernel). It turns out that $\mathcal{C}^{-2}$ is a sesquiholomorphic polynomial which vanishes nowhere on $D \times \mathrm{Cl}(D)$. Hence, $\mathcal{C}$ extends to a sesquiholomorphic function on a neighbourhood of $D \times \mathrm{Cl}(D)$.

Korányi: Define the Poisson–Szegő kernel $\mathcal{P}$ so that

$$\mathcal{P}\colon D \times \mathrm{b}D \ni (z,\zeta) \mapsto \frac{|\mathcal{C}(z,\zeta)|^2}{\mathcal{C}(z,z)}.$$

Then:

- $\mathcal{P}$ is continuous and bounded;
- $\int_{\mathrm{b}D} \mathcal{P}(z,\zeta)\,\mathrm{d}\beta_{\mathrm{b}D}(\zeta) = 1$ for every $z \in D$;
- $(\mathcal{P}\mu)_\rho \cdot \beta_{\mathrm{b}D} \to \mu$ vaguely for $\rho \to 1^-$, where $(\mathcal{P}\mu)_\rho(\zeta) = \int_{\mathrm{b}D} \mathcal{P}(\rho\zeta,\zeta')\,\mathrm{d}\beta_{\mathrm{b}D}(\zeta')$, for every Radon measure μ on $\mathrm{b}D$.

A real Radon measure μ on $\mathrm{b}D$ is pluriharmonic if the following equivalent conditions hold:

- $\mathcal{P}\mu$ is pluriharmonic (i.e., the real part of a holomorphic function on D);
- $\mathcal{P}\mu = \mathrm{Re}\,\mathcal{H}\mu = \mathrm{Re}\int_{\mathrm{b}D}(2\mathcal{C}_\zeta - 1)\,\mathrm{d}\mu(\zeta)$;
- μ is in the vague closure of the space of pluriharmonic polynomials;
- $\int_{\mathrm{b}D} P\,\mathrm{d}\mu = 0$ for every polynomial P such that $\int_{\mathrm{b}D} PQ\,\mathrm{d}\beta_{\mathrm{b}D} = 0$ for every pluriharmonic polynomial Q.

(Korányi–Pukánszky) For every positive pluriharmonic function $f\colon D \to \mathbb{R}$ there is a unique positive pluriharmonic measure μ on $\mathrm{b}D$ such that

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Aleksandrov, Aleksandrov–Doubtsov, C.: Let μ be a pluriharmonic measure on $\mathrm{b}D$, and let $\widehat{\mathrm{b}D}$ be the quotient of $\mathrm{b}D$ by the action of $\mathbb{T}$; let $\pi\colon \mathrm{b}D \to \widehat{\mathrm{b}D}$ be the canonical projection and set $\beta_{\widehat{\mathrm{b}D}} := \pi_*(\beta_{\mathrm{b}D})$.
Then, μ has a 'disintegration' $(\mu_\xi)_{\xi\in\widehat{\mathrm{b}D}}$ relative to $\beta_{\widehat{\mathrm{b}D}}$, that is, μ_ξ is a Radon measure on $\mathrm{b}D$ concentrated on $\pi^{-1}(\xi)$ and

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In addition,

$$(\mathcal{P}\mu)(z) = \int_{\mathrm{b}D} \frac{1-|z|^2}{|z-\zeta|^2}\,\mathrm{d}\mu_\xi(\zeta)$$

for every $z \in \mathbb{D}_\xi = \mathbb{D}\pi^{-1}(\xi)$ and for almost every $\xi \in \widehat{\mathrm{b}D}$.
Further, if μ is positive, then (μ_ξ) may be chosen to be vaguely continuous (and the above identity holds for every $\xi \in \widehat{\mathrm{b}D}$).

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Sketch of the proof

Take a pluriharmonic measure μ on $\mathrm{b}D$ and set $f = \mathcal{P}\mu$. Then,

$$\int_{\widehat{\mathrm{b}D}} \sup_{\rho\in(0,1)} \|f\|_{L^1(\rho\pi^{-1}(\xi))} \,\mathrm{d}\beta_{\widehat{\mathrm{b}D}}(\xi) = \sup_{\rho\in(0,1)} \int_{\widehat{\mathrm{b}D}} \|f\|_{L^1(\rho\pi^{-1}(\xi))} \,\mathrm{d}\beta_{\widehat{\mathrm{b}D}}(\xi) = \lim_{\rho\to 1^-} \|f\|_{L^1(\rho \mathrm{b}D)} = \|\mu\|_{\mathcal{M}^1(\mathrm{b}D)}$$

since f is harmonic on the disc $\mathbb{D}\pi^{-1}(\xi)$ for every $\xi \in \widehat{\mathrm{b}D}$. Thus, the restriction of f to the disc $\mathbb{D}\pi^{-1}(\xi)$ belongs to the *harmonic* Hardy space H^1 for almost every ξ, so that μ_ξ exists and

$$(\mathcal{P}\mu)(z) = \int_{\mathrm{b}D} \frac{1-|z|^2}{|z-\zeta|^2} \,\mathrm{d}\mu_\xi(\zeta).$$

Vague convergence then leads to the disintegration formula.

C.: If μ is a pluriharmonic measure on $\mathrm{b}D$ and $m = \dim \mathrm{b}D$, then μ is absolutely continuous with respect to $\mathcal{H}^{m-1}$.

Aleksandrov, Aleksandrov–Doubtsov, C.: If μ is a pluriharmonic measure on $\mathrm{b}D$, then

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Poltoratski, Aleksandrov, Aleksandrov–Doubtsov, C.: If μ is a pluriharmonic measure on $\mathrm{b}D$, then

$$\lim_{y \to +\infty} \pi y \chi_{\{\zeta \in \mathrm{b}D \colon \mathcal{C}\mu(\zeta) > y\}} \cdot \beta_{\mathrm{b}D} = |\mu^s|$$

vaguely, where μ^s denotes the singular part of μ (with respect to $\beta_{\mathrm{b}D}$).

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Intermezzo: Unbounded Realizations

Every bounded symmetric domain is biholomorphic to a Siegel domain $\mathscr{D} = \{(\zeta, z) \colon \operatorname{Im} z - Q(\zeta) \in \Omega\}$ of type II, with Šilov boundary $\{(\zeta, z) \colon \operatorname{Im} z = Q(\zeta)\}$. For example:

- the unit ball B of $\mathbb{C}^{n+1}$ is biholomorphic to the Siegel upper half-space $U = \{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C} \colon \operatorname{Im} z > |\zeta|^2\}$ through the Cayley transform $(\zeta, z) \mapsto \left(\frac{2\zeta}{1-z}, i\frac{1+z}{1-z}\right)$;
- the polydisc $\mathbb{D}^n$ is biholomorphic to $\mathbb{C}_+^n$, where $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, through the Cayley transform, $(z_j) \mapsto \left(i\frac{1+z_j}{1-z_j}\right)$;
- the unit ball of $M_n(\mathbb{C})$ is biholomorphic to $\{A + iB \in M_n(\mathbb{C}) \colon A, B \in M_n(\mathbb{C}), A = A^*, B = B^*, B > 0\}$ through the Cayley transform $A \mapsto i(I - A)^{-1}(I + A)$.

One may define a Hardy space on $\mathscr{D}$ and an associated Poisson–Szegő kernel $\mathscr{P}$, and define Poisson-summable measures on $\mathrm{b}\mathscr{D}$ accordingly:

$$H^2(\mathscr{D}) = \left\{ f \in \mathrm{Hol}(\mathscr{D}) \colon \sup_{h \in \Omega} \int_{\mathrm{b}\mathscr{D}} |f(\zeta, z + ih)|^2 \, \mathrm{d}(\zeta, z) < +\infty \right\},$$

with Cauchy–Szegő kernel $\mathscr{C}$ and

$$\mathscr{P}((\zeta, z), (\zeta', z')) = \frac{|\mathscr{C}((\zeta, z), (\zeta', z'))|^2}{\mathscr{C}((\zeta, z)(\zeta, z))}.$$

For example, $\mathscr{P}(z, z') = \prod_j \frac{\mathrm{Im}\, z_j}{|z - z'|^2}$ when $\mathscr{D} = \mathbb{D}^n$.

Definition

A Poisson-summable measure μ on $\mathrm{b}\mathscr{D}$ is pluriharmonic if $\mathscr{P}\mu$ is pluriharmonic.

Let $f\colon \mathscr{D} \to [0,+\infty)$ be a plurihamonic function. Then, the vague limit μ of the restrictions of $f(\,\cdot\,, \cdot + ih)$ to $\mathrm{b}\mathscr{D}$, for $h \to 0$, exists, and is the largest Poisson-summable positive measure such that $\mathscr{P}\mu \leqslant f$.

- [C] if $\mathscr{D}$ is the Siegel upper half-space, then $f = \mathscr{P}\mu$;
- [Luger–Nedic] if $\mathscr{D} = \mathbb{C}^n_+$, then $f(z) = \mathscr{P}\mu(z) + \sum_j a_j \mathrm{Im}\, z_j$ for some $a_1, \dots, a_n \geqslant 0$;
- [C] if $\mathscr{D}$ corresponds to the unit ball in $M_n(\mathbb{C})$, then μ need *not* be pluriharmonic.

Clark Measures

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Denote by μ_α^s its singular part with respect to $\beta_{\mathrm{b}D}$.

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- ('Aleksandrov's disintegration theorem')

$$\int_{\mathbb{T}} \mu_\alpha \, \mathrm{d}\beta_{\mathbb{T}}(\alpha) = \beta_{\mathrm{b}D};$$

- if φ is inner (i.e., $|\varphi_1(\zeta)| = 1$ for a.e. $\zeta \in \mathrm{b}D$), then (μ_α) is a disintegration of $\beta_{\mathrm{b}D}$ relative to $\beta_{\mathbb{T}}$;
- if φ is inner and $\varphi(0) = 0$, then $(\varphi_1)_*(\beta_{\mathrm{b}D}) = \beta_{\mathbb{T}}$;
- if $\varphi(0) = 0$, then

$$\int_{\mathrm{b}D} \overline{P} \, \mathrm{d}\mu_\alpha = \sum_{h=1}^{k} \overline{\alpha}^h \int_{\mathrm{b}D} \varphi^h \overline{P} \, \mathrm{d}\beta_{\mathrm{b}D}$$

  for every homogeneous (holomorphic) polynomial P of degree $k \in \mathbb{N}$.

The mapping $\mathcal{C}_\alpha \colon L^2(\mu_\alpha) \to H^2(D)$,

$$\mathcal{C}_\alpha(f)(z) = (1 - \overline{\alpha}\varphi(z)) \int_{\mathrm{b}D} f(\zeta)\mathcal{C}_\zeta(z)\,\mathrm{d}\mu_\alpha(\zeta),$$

is contractive. If $H^2(\mu_\alpha)$ denotes the closure of the space of *holomorphic* polynomials in $L^2(\mu_\alpha)$, then $\ker \mathcal{C}_\alpha = H^2(\mu_\alpha)^\perp$.

When $D = U$, the unit ball in $\mathbb{C}^n$, then:

- Doubtsov: if μ is a pluriharmonic measure on $\mathrm{b}D$ which is singular with respect to $\beta_{\mathrm{b}D}$, then μ is 'totally singular', that is, μ is singular with respect to any probability measure ν on $\mathrm{b}D$ such that $\int_{\mathrm{b}D} P\,\mathrm{d}\nu = P(z)$ for every holomorphic polynomial P (and some fixed $z \in D$);
- Aleksandrov–Doubtsov: if φ is inner, then μ_α is totally singular and $H^2(\mu_\alpha) = L^2(\mu_\alpha)$.

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[Bickel–Cima–Sola, Anderson–Bergqvist–Bickel–Cima–Sola, C.]
Let $\varphi\colon D \to \mathbb{D}$ be a rational inner function. Then,

$$\int_{\mathrm{b}D} f \,\mathrm{d}\mu_\alpha = \int_{\mathrm{b}D\cap\varphi^{-1}(\alpha)} \frac{c}{|\nabla\varphi(z)|} f(z)\,\mathrm{d}\mathcal{H}^{m-1}(z),$$

where $m = \dim \mathrm{b}D$.
In addition, if $\varphi = p/q$, with p and q coprime, then the closure of $\mathrm{b}D \cap \varphi^{-1}(\alpha)$ is the real algebraic set $\{\zeta \in \mathrm{b}D\colon (p - \alpha q)(\zeta) = 0\}$.

[Bickel–Cima–Sola, Anderson–Bergqvist–Bickel–Cima–Sola, C.]
Let $\varphi\colon \mathbb{D}^n \to \mathbb{D}$ be a rational inner function. Then, $H^2(\mu_\alpha) \neq L^2(\mu_\alpha)$ if and only if the closure of $\mathrm{b}D \cap \varphi^{-1}(\alpha)$ contains a set of the form $\mathbb{T} \times V$, up to a permutation of the coordinates, where V is a real algebraic hypersurface in $\mathbb{T}^{n-1}$.

Example

If $\varphi\colon \mathbb{D}^2 \to \mathbb{D}$ and

$$\varphi(z, z') = \frac{2zz' - z - z'}{2 - z - z'},$$

then φ is a rational inner function and $H^2(\mu_\alpha) = L^2(\mu_\alpha)$ if and only if $\alpha \neq -1$.

Proof (case $n = 2$)

Assume that $\mathbb{T}^2 \cap \varphi^{-1}(\alpha)$ does not contain sets of the form $\mathbb{T} \times \{ \alpha' \}$ or $\{ \alpha' \} \times \mathbb{T}$, $\alpha' \in \mathbb{T}$. Setting $\varphi = p/q$ with p and q coprime, observe that

$$(p - \alpha q)(z, z') = q_1(z') + z p_1(z, z')$$

for some unique p_1, q_1. Observe that, if $z' \in \mathbb{D}$ and $q_1(z') = 0$, then $(p - \alpha q)(0, z') = 0$, that is, $\varphi(0, z') = \alpha \notin \mathbb{D}$: absurd. If $\alpha' \in \mathbb{T}$ and $q_1(\alpha') = 0$, then $\varphi(\,\cdot\,, \alpha')$ is a well-defined non-constant (by the assumption) rational inner function on $\mathbb{D}$, and takes the value α at 0: absurd. Thus,

$$\overline{z} = -\frac{p_1(z, z')}{q_1(z')}$$

for every $z \in \mathbb{T}^2 \cap \varphi^{-1}(\alpha)$. Since p_1/q_1 may be uniformly approximated with holomorphic polynomials on $\mathbb{T}^2$, this proves that $\overline{z} \in H^2(\mu_\alpha)$. In a similar way on shows that $\overline{z'} \in H^2(\mu_\alpha)$, and then that all polynomials (holomorphic or not) belong to $H^2(\mu_\alpha)$. Hence, $H^2(\mu_\alpha) = L^2(\mu_\alpha)$.

Continuation of the Proof

Conversely, assume that $\{\alpha'\}\times\mathbb{T}$ is contained in the closure of $\mathbb{T}^2\cap\varphi^{-1}(\alpha)$ for some $\alpha'\in\mathbb{T}$. Then, it is possible to prove that $|\nabla\varphi|$ is constant on $\{\alpha'\}\times\mathbb{T}$, so that $\mu_\alpha \geqslant c\chi_{\{\alpha'\}\times\mathbb{T}}\cdot\mathcal{H}^1$. Assuming by contradiction that $H^2(\mu_\alpha)=L^2(\mu_\alpha)$ and taking a sequence (p_j) of holomorphic polynomials on $\mathbb{D}^2$ which converge to $\overline{z'}$ in $L^2(\mu_\alpha)$, this proves that $p_j(\alpha',\,\cdot\,)$ is a sequence of holomorphic polynomials on $\mathbb{D}$ which converge to $\overline{z'}$ in $L^2(\chi_\mathbb{T}\cdot\mathcal{H}^1)$: absurd.

Thank you for your attention!