Regularity theory for mixed local and nonlocal p-Laplace equations

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Outline

Main equation

Mixed local and nonlocal p-Laplace equation

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 0 < s < 1,\ 1 < p < \infty,$$

where

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

and

$$(-\Delta_p)^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy.$$

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p-Laplace equation

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$

Weak Solution

- $u : \Omega \to \mathbb{R}$.
- $$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi \, dx = 0, \quad \forall \phi \in C_c^\infty(\Omega).$$

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Fractional p-Laplace equation

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For every $\phi \in C_c^\infty(\Omega)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+sp}}\,dx\,dy = 0.$$

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- $$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+sp}}\,dx\,dy + 2\int_{x\in\Omega}\int_{y\in\mathbb{R}^n\setminus\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))\phi(x)}{|x-y|^{n+sp}}\,dx\,dy = 0.$$
- First integral: Local.
- Second integral: Nonlocal.

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Mixed local and nonlocal case

Mixed equation

$$-\Delta_p u + (-\Delta_p)^s u = 0 \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 < s < 1 < p < \infty$.

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To establish regularity properties of weak solutions which may change sign for the mixed equation:

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Known results

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$$-\Delta u + (-\Delta)^s u = 0, \quad 0 < s < 1.$$

- **Foondun, Chen-Kim-Song-Vondraček:** Harnack inequality for globally nonnegative solutions

$$\sup_{B_r} u \leq c \inf_{B_r} u.$$

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Nonlinear mixed case

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 0 < s < 1,\ 1 < p < \infty.$$

Drawbacks

- Lack of analytic approach even for the linear case.
- Lack of algebraic inequalities.
- Nonlinearity and the mixed behavior.

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p-Laplace equation

$$\Delta_p u = 0, \quad 1 < p < \infty.$$

Local boundedness:

$$\sup_{B_r} u \leq c\Big(\frac{1}{|B_r|}\int_{B_r} u^p\,dx\Big)^{\frac{1}{p}}.$$

Weak Harnack's inequality:

$$\Big(\frac{1}{|B_r|}\int_{B_r} u^l\,dx\Big)^{\frac{1}{l}} \leq c\inf_{B_r} u,\ \ l > 0.$$

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- Landkof: Harnack inequality for **globally nonnegative ($u \geq 0$ in $\mathbb{R}^n$) solutions**

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- Kassmann: Harnack inequality fails for sign changing solutions.
- If $u \geq 0$ in $B_R(x_0)$, then for any $0 < r < R$,

$$\sup_{B_r} u \leq c \inf_{B_r} u + c\left(\frac{r}{R}\right)^{2s} \mathrm{Tail}_{2s}(u_-; x_0, R).$$

- $u_- = \max\{-u, 0\}$.
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$$\mathrm{Tail}_{2s}(u, x_0, R) = R^{2s} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{|u(y)|}{|y - x_0|^{n+2s}}\, dy.$$

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$$(-\Delta)^s u = 0, \quad 0 < s < 1.$$

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- Developed regularity theory for the equation

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 0 < s < 1 < p < \infty.$$

- Approach is purely analytic.
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$$\sup_{B_{\frac{r}{2}}(x_0)} u \leq c \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u_+^p \, dx \right)^{\frac{1}{p}} + c \, \mathrm{Tail}_p\left(u_+; x_0, \tfrac{r}{2}\right).$$

- $u_+ = \max\{u, 0\}$.

Main results

Local boundedness of solutions

- Let u be a weak solution of

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Key steps to prove local boundedness

Key steps

- Energy estimate.
- Algebraic inequality.
- Sobolev inequality.
- Iteration lemma.

Sketch of the proof

Energy estimate

For $w = (u-k)_+$, $k \in \mathbb{R}$,

$$\int_{B_r(x_0)} \psi^p |\nabla w|^p \, dx + \int_{B_r(x_0)} \int_{B_r(x_0)} \frac{|w(x)\psi(x) - w(y)\psi(y)|^p}{|x-y|^{n+sp}} \, dx \, dy$$
$$\leq C \Bigg(\int_{B_r(x_0)} w^p |\nabla \psi|^p \, dx$$
$$+ \int_{B_r(x_0)} \int_{B_r(x_0)} \frac{\max\{w(x), w(y)\}^p |\psi(x) - \psi(y)|^p}{|x-y|^{n+sp}} \, dx \, dy$$
$$+ \sup_{x \in \mathrm{supp}\psi} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{w(y)^{p-1}}{|x-y|^{n+sp}} \, dy \cdot \int_{B_r(x_0)} w \psi^p \, dx \Bigg),$$

where $\psi \in C_c^\infty(B_r(x_0))$ is a nonnegative function.

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Algebraic inequality, De Castro-Kuusi-Palatucci

- Let $p \geq 1$ and $\epsilon \in (0, 1]$.
- Then for every $a, b \in \mathbb{R}^n$, we have

$$|a|^p \leq |b|^p + c(p)\epsilon|b|^p + \big(1 + c(p)\epsilon\big)\epsilon^{1-p}|a - b|^p,$$

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Iteration lemma

Let $(Y_j)_{j=0}^{\infty}$ be a sequence of positive real numbers such that

$$Y_0 \leq c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}$$

and

$$Y_{j+1} \leq c_0 b^j Y_j^{1+\beta}, \quad j = 0, 1, 2, \ldots$$

for some constants $c_0, b > 0$ and $\beta > 0$. Then $\lim_{j\to\infty} Y_j = 0$.

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Main results

Weak Harnack inequality for solutions

- Let u be a weak solution of

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 0 < s < 1 < p < \infty.$$

- $u \geq 0$ in $B_R(x_0) \subset \Omega$.
- For any $0 < r < R$,

$$\left(\frac{1}{|B_{\frac{r}{2}}(x_0)|}\int_{B_{\frac{r}{2}}(x_0)} u^l\,dx\right)^{\frac{1}{l}} \leq c\inf_{B_r(x_0)} u + c\left(\frac{r}{R}\right)^{\frac{p}{p-1}}\mathrm{Tail}_p(u_-;x_0,R), \ l > 0.$$

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- Expansion of positivity.
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Expansion of positivity

- Let u be a weak solution of

$$-\Delta_p u + (-\Delta_p)^s u = 0, \quad 0 < s < 1, \quad 1 < p < \infty,$$

such that $u \geq 0$ in $B_R(x_0) \subset \Omega$.

- Assume $k \geq 0$ and there exists $\tau \in (0, 1]$ such that

$$\Big|B_r(x_0) \cap \{u \geq k\}\Big| \geq \tau |B_r(x_0)|, \tag{1}$$

for some $r \in (0, 1]$ with $0 < r < \frac{R}{16}$.

- There exists a constant $\delta \in (0, \frac{1}{4})$ such that

$$\inf_{B_{4r}(x_0)} u \geq \delta k - \left(\frac{r}{R}\right)^{\frac{p}{p-1}} \mathrm{Tail}_p(u_-; x_0, R). \tag{2}$$

Iteration lemma

- Let $0 \le T_0 \le t \le T_1$ and assume that $f : [T_1, T_2] \to [0, \infty)$ is a nonnegative bounded function.
- Suppose that for $T_0 \le t < s \le T_1$, we have

$$f(t) \le A(s-t)^{-\alpha} + B + \theta f(s), \tag{3}$$

  where A, B, α, θ are nonegative constants and $\theta < 1$.
- Then there exists a constant $c = c(\alpha, \theta)$ such that for every ρ, R and $T_0 \le \rho < R \le T_1$, we have

$$f(\rho) \le c(A(R-\rho)^{-\alpha} + B). \tag{4}$$

Recent developments

$$-\Delta_p u + (-\Delta_p)^s u = f, \quad 0 < s < 1 < p < \infty.$$

- Biagi-Dipierro-Valdinoci-Vecchi: Local boundedness among other results.
- De Filippis-Mingione: Gradient regularity.
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$$-\Delta_p u + (-\Delta_p)^s u = f(x)u^{-\delta}, \quad 0 < s < 1 < p < \infty, \quad \delta > 0.$$

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$$u_t - \Delta_p u + (-\Delta_p)^s u = f, \quad 0 < s < 1 < p < \infty.$$

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Thank You for Your Attention!