Statistics of Continued fractions II: homogeneous dynamics

Anish Ghosh

(Joint work with Gaurav Aggarwal)

Continued Fractions

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots$ are natural numbers.

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$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \frac{1}{a_k}}}} \in \mathbb{Q}$$

The Lévy-Khintchine theorem

For Lebesgue almost every θ:

$$\lim_{n\to\infty} q_k^{1/k} = \exp \frac{\pi^2}{12 \log 2}$$

- Aleksandr Khintchine (1936) proved the existence of the limit.
- Paul Lévy (1936) computed the value of the limit.

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Higher Dimensional Analogues

One dimensional convergents are best approximations!

One dimensional convergents

For all $0 < q < q_k$ and $p \in \mathbb{N}$, we have

$$|\theta q_k - p_k| < |\theta q - p|.$$

One dimensional convergents

An integer vector $(p,q) \in \mathbb{Z} \times \mathbb{N}$ is a best approximation of θ if and only if there is no $(p',q') \in \mathbb{Z} \times \mathbb{N}$ such that

$$|p' + \theta q'| \leq |p + \theta q|,$$
$$|q'| < |q|.$$

A natural Higher Dimensional Analogue

Best Approximation

Given $\theta \in M_{m\times n}(\mathbb{R})$, then $(p,q) \in \mathbb{Z}^m \times \mathbb{Z}^n$ is called a best approximation of θ if there is no integral solution $(p',q') \in \mathbb{Z}^m \times \mathbb{Z}^n$ other than $\pm(p,q)$ satisfying

$$\|p' + \theta q'\| \leq \|p + \theta q\|$$
$$0 < \|q'\| \leq \|q\|.$$

Remark

The definition depends on the choice of the norms in $\mathbb{R}^m$ and $\mathbb{R}^n$.

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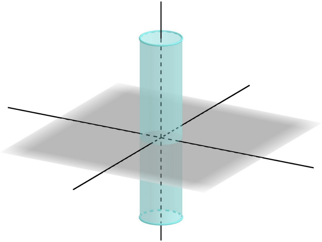
Best approximation

Given $\theta \in M_{m\times n}(\mathbb{R})$, then $(p,q) \in \mathbb{Z}^m \times \mathbb{Z}^n$ is called a best approximation of θ if there are no primitive vectors of the lattice $\Lambda_\theta = \begin{pmatrix} I_m & \theta \\ & 1 \end{pmatrix} \mathbb{Z}^{m+1}$ in the cylinder



other than $(p+\theta q, q)$ and $(-p-\theta q, -q)$.

Best Approximation

Cheung-Chevallier, Ann. Sci. École Norm. Sup. (2024)

Suppose the norms $\|.\|_m$ and $\|.\|_n$ are **standard Euclidean norms**.
For **Lebesgue**-a.e. θ we have:
the sequence (p_k, q_k) of best approximations of θ corresponding to the Euclidean norm satisfy:
There exists a constant $L_{m,n}$ such that

$$\frac{1}{k}\log(\|q_k\|) \to L_{m,n}$$

Nondegenerate curves

Aggarwal-Ghosh, 2024

For a.e. θ on a nondegenerate curve in $\mathbb{R}^n$, we have:
the sequence (p_k, q_k) of best approximations of θ corresponding to the Euclidean norm satisfy:
There exists a constant L_n such that

$$\frac{1}{k}\log(\|q_k\|) \to L_n$$

This is not the only generalization to higher dimensions.

This is one end of a spectrum of generalized definitions of best approximation.

One dimensional convergents

An integer vector $(p,q) \in \mathbb{Z} \times \mathbb{N}$ is a best approximation of θ if and only if there are no non-zero vectors of the lattice

$$\Lambda_\theta = \begin{pmatrix} 1 & \theta \\ & 1 \end{pmatrix} \mathbb{Z}^2$$

in the rectangle

$$[-p-\theta q, p+\theta q] \times [-q, q],$$

other than $(p+\theta q, q)$ and $(-p-\theta q, -q)$.

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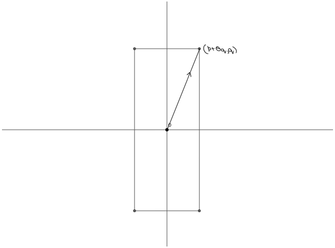

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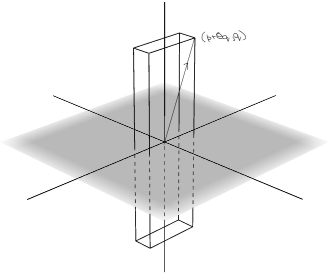
Higher dimensional convergents

An integer vector $(p,q) \in \mathbb{Z}^m \times \mathbb{N}$ is a best approximate of $\theta \in M_{m\times 1}(\mathbb{R})$ if there are no primitive vectors of the lattice $\Lambda_\theta = \begin{pmatrix} I_m & \theta \\ & 1 \end{pmatrix} \mathbb{Z}^{m+1}$ in the hyper-cuboid



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An integer vector $(p, q) \in \mathbb{Z}^m \times \mathbb{N}$ is a best approximate of $\theta \in M_{m\times 1}(\mathbb{R})$ if there are no primitive vectors of the lattice

$$\Lambda_\theta = \begin{pmatrix} I_m & \theta \\ & 1 \end{pmatrix} \mathbb{Z}^{m+1}$$

in the hyper-cuboid

$$[-p_1 - \theta_1 q, p_1 + \theta_1 q] \times \ldots \times [-p_m - \theta_m q, p_m + \theta_m q] \times [-q, q],$$

other than $(p + \theta q, q)$ and $(-p - \theta q, -q)$.

Cheung's Conjecture

For Lebesgue a.e. $\theta \in \mathbb{R}^m$, if (p_l, q_l) denotes the best approximates of θ, then the limit

$$\lim_{l \to \infty} \frac{(\log q_l)^m}{l}$$

exists and is independent of θ.

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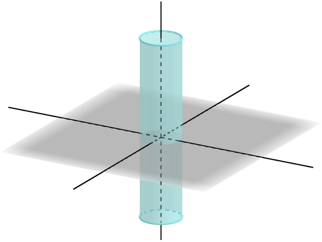
[Aggarwal-Ghosh, 2024]: Cheung's Conjecture holds.

Comparison



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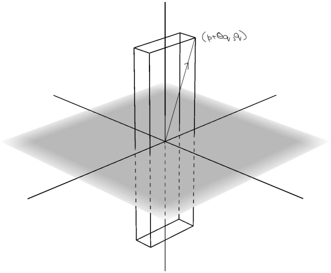

Cheung's conjecture case

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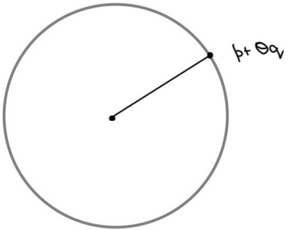

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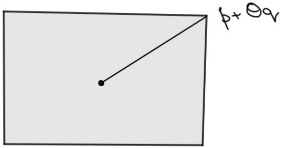

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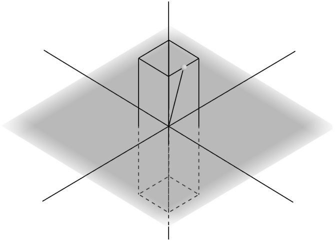

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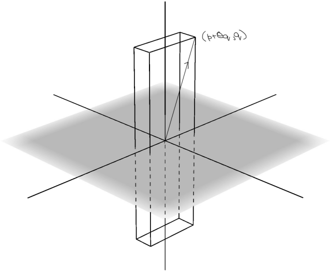

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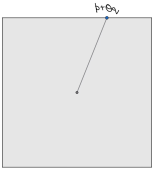

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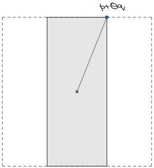

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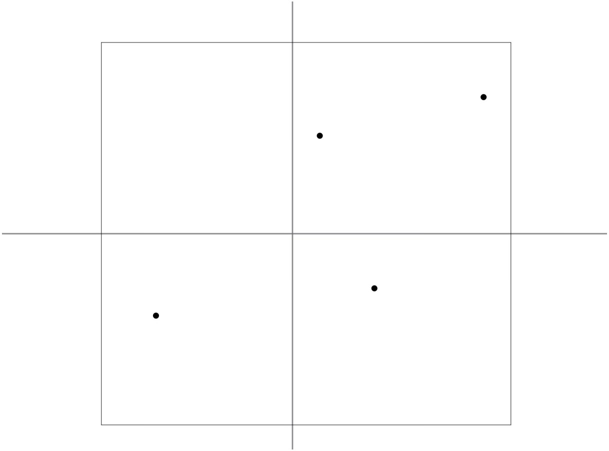

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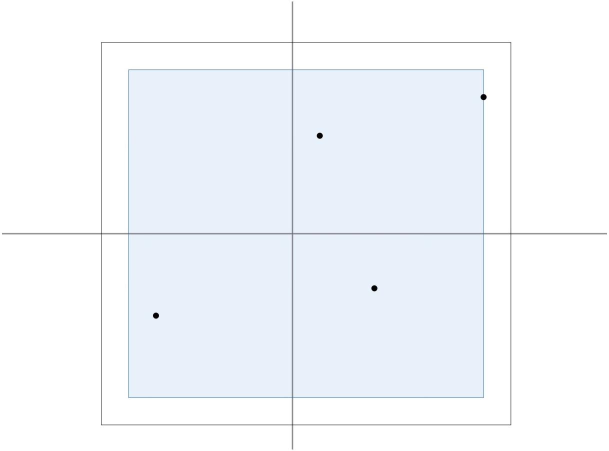

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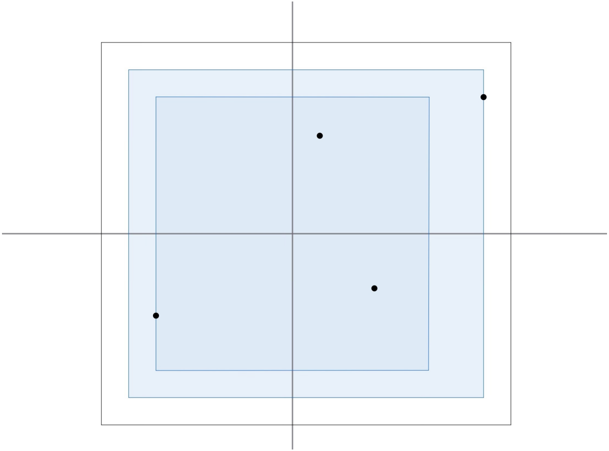

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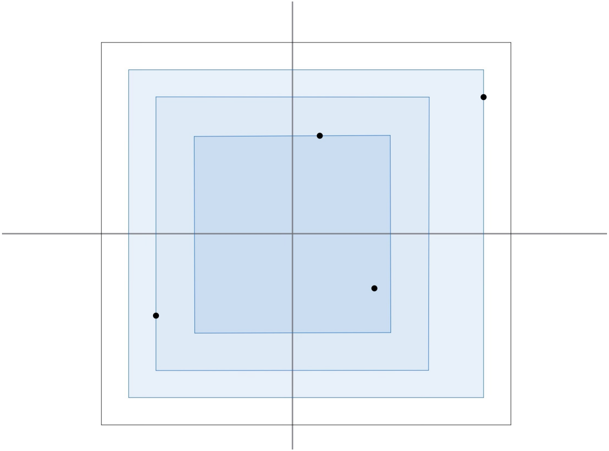

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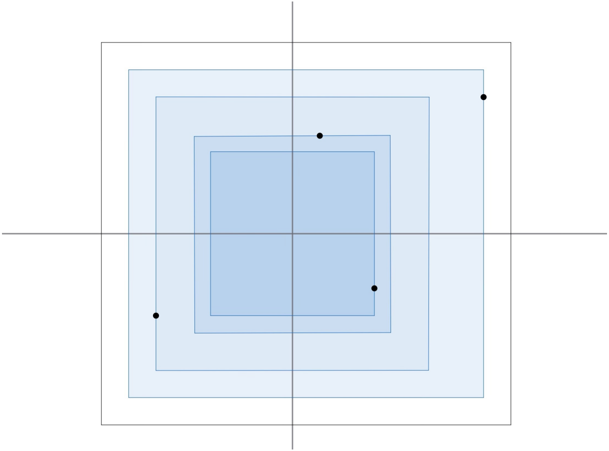

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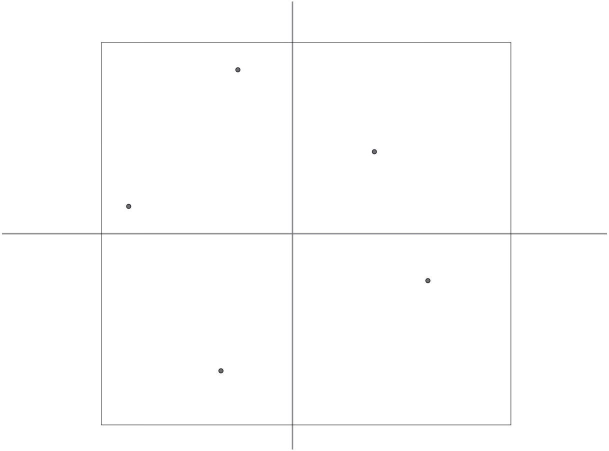

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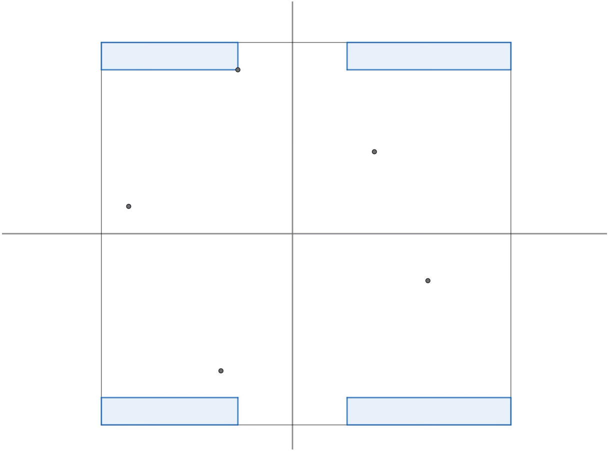

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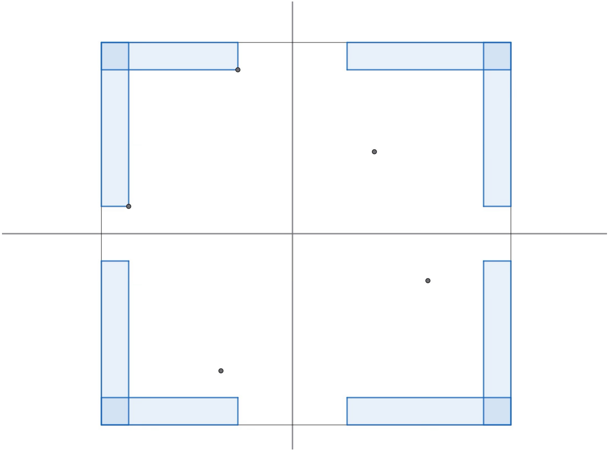

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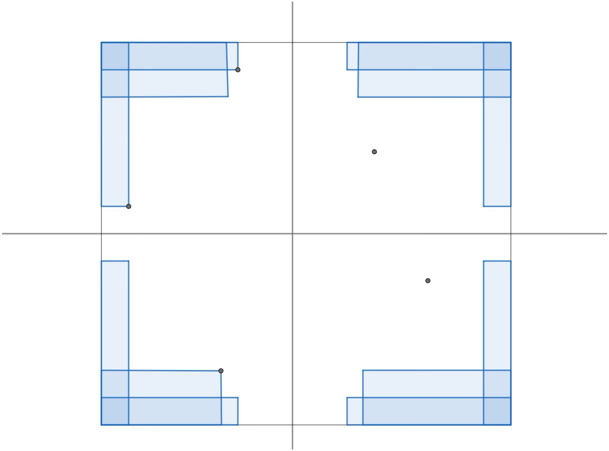

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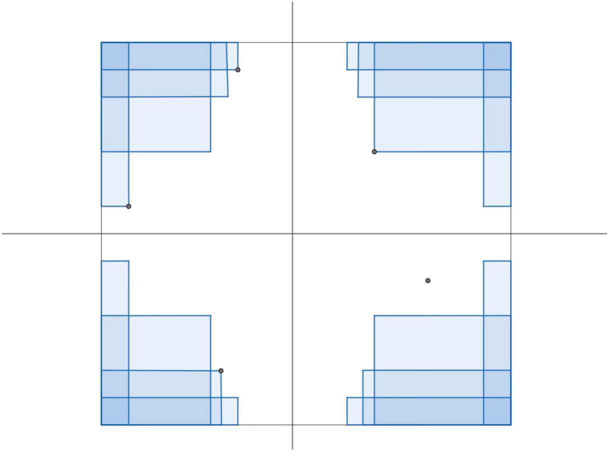

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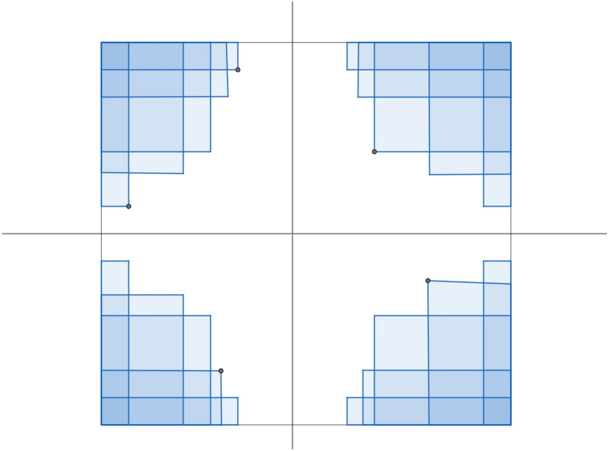

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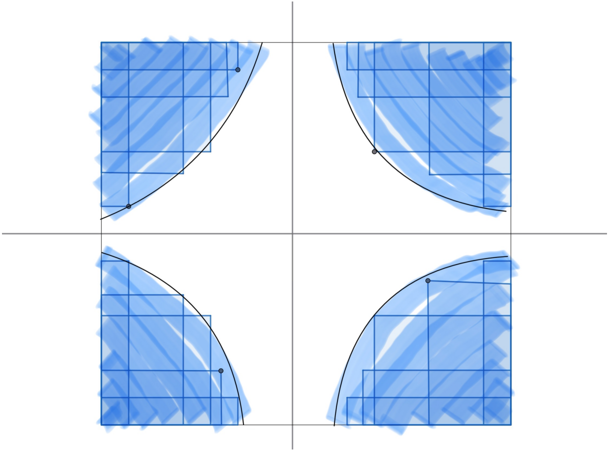

Results

- Lévy-Khintchine theorem
  - For self similar fractals
  - For non-degenerate curves
  - For general norms
- Cheung's conjecture
- Joint equidistribution of best approximates

Notation and Interpretation

- For $\theta \in \mathrm{M}_{m\times n}(\mathbb{R})$, define

$$\mathscr{N}(\theta, T)$$

as the number of best approximates (p, q) of θ with $\|q\| \leq e^{T}$.

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since $(p_1, q_1), \ldots, (p_k, q_k)$ are exactly the best approximates with $\|q_j\| \leq \|q_k\|$.

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- The number of best approximates (p, q) with $\|q\|$ less than or equal to e^T grows linearly in T.

Main Theorem

- For A in co-null subset of $M_{m\times n}(\mathbb{R})$, the number of best approximants (p,q) of A with $\|q\| < T$ and satisfying the following
  - $\left(\frac{p_1 + A_1 q}{\|p_1 + A_1 q\|}, \dots, \frac{q_r}{\|q_r\|}\right) \in X_1$
  - The error $\|q_1\|^{n_1} \dots \|q_r\|^{n_r} \|p_1 + A_1 q\|^{m_1} \dots \|p_r + A_r q_r\|^{m_k} \in X_2$
  - The lattice $\Lambda(A,p,q) \in X_3$
  - $(p,q) = I \mod N$ (or $(p,q) \in X_4 \subset \hat{\mathbb{Z}}^{m+n}$)

  are asymptotically $\sim T^{k+r-1}\nu(X_1 \times X_2 \times X_3 \times X_4)$.
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Corollary

For Lebesgue almost every $\theta \in \mathrm{M}_{m\times n}(\mathbb{R})$, the sequence $(p_l, q_l) \in \mathbb{Z}^m \times \mathbb{Z}^n$ of best approximations to θ, ordered by increasing $\|q_l\|$, satisfies

$$\lim_{l\to\infty} \frac{(\log \|q_l\|)^{k+r-1}}{l} = \left(\frac{c_{k+r-1}(n_1, \ldots, n_r)}{(k+r-1)!\,\zeta(d)} \cdot \tilde{\nu}^j(\mathscr{Z}_j) \right)^{-1}. \qquad (1)$$

- Cheung's conjecture follows in the special case $n = r = 1$, $k = m$.
- In fact, we obtain a very general *Lévy-Khintchine-type theorem with constraints*, where one can impose congruence, directional, or size restrictions on the approximates.

Dimension one

- Results by LeVeque (1958) and Philipp (1967-1970)

- Philipp and Stackelberg (1969): Law of iterated logarithm, and gave sharp bound on error

$$\log q_k = \gamma k + O(\sqrt{k \log\log k})$$

- Ibragimov, Misevicius, Morita and Vallée (1961-1997): Studied the distribution of

$$\frac{\log q_k - \gamma k}{\sqrt{k}},$$

and proved a Central limit theorem, with an explicit convergence rate.

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Dimension one

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Question

- Analogues for middle third Cantor set?

- Analogues in higher dimension?

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Suppose μ is either

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Theorem 1 (Aggarwal-Ghosh,2025)

There exists $\gamma > 0$ such that the following holds. For any $\varepsilon > 0$, we have

$$\mathcal{N}(\theta, T) = \gamma T + O\left(T^{1/2} \log^{\frac{3}{2}+\varepsilon} T\right), \tag{2}$$

for μ-almost every $\theta \in \mathrm{M}_{m\times n}(\mathbb{R})$.

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Theorem 1 (Aggarwal-Ghosh,2025)

There exists a $\gamma, \sigma > 0$ such that for every $\xi \in \mathbb{R}$, we have

$$\mu\left(\left\{\theta \in \mathrm{M}_{m\times n}(\mathbb{R}) : \frac{\mathcal{N}(\theta, T) - \gamma T}{T^{1/2}} < \xi\right\}\right) \longrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx,$$

as $T \to \infty$.

Remarks

- For dimension 1 and Lebesgue measure, the results are weaker than known results.

- For ε-approximation in place of best approximations
  - Effective counting by Schmidt, Gallagher
  - Central limit theorem by Dolgopyat, Fayad, Vinogradov and Björklund, Gorodnik

- The above theorem holds for any probability measure μ on $\mathrm{M}_{m\times n}(\mathbb{R})$ satisfying Condition (EMEI)

- The value of γ and σ depends only on m, n and choice of norms $\|\cdot\|$, and is given explicitly in terms of integral of measurable function on the space of uni-modular lattices in $\mathbb{R}^{m+n}$.

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Condition (EMEI)

A probability measure μ on $\mathrm{M}_{m\times n}(\mathbb{R})$ will be called to satisfy Condition (EMEI) (short for *Effective Multi-equidistribution for Identity coset under diagonal flow*

$$a_t = \begin{pmatrix} e^{t/m} I_m & \\ & e^{-t/n} I_n \end{pmatrix}$$

if it satisfies the following properties:

- μ is compactly supported,
- For all $F_0 \in C^\infty(\mathrm{M}_{m\times n}(\mathbb{R}))$, $F_1, \dots, F_r \in C_c^\infty(\mathscr{X})$ and $t_1, \dots, t_r > 0$, we have

$$\int_{\mathrm{M}_{m\times n}(\mathbb{R})} F_0(\theta) \left(\prod_{i=1}^{r} F_i(a_{t_i} u(\theta)\Gamma) \right) d\mu(\theta) = \mu(F_0)\mu_{\mathscr{X}}(F_1) \cdots \mu_{\mathscr{X}}(F_r) + O_r\left(e^{-\delta D(t_1,\dots,t_r)} \|F_0\|_{C^k} \prod_{i=1}^{r} \|F_i\|_{C^k} \right)$$

where

$$D(t_1, \dots, t_r) = \min\{t_i, |t_i - t_j| : 1 \le i, j \le r, i \ne j\}.$$

Condition (EMEI) for fractal measures

- Kleinbock-Margulis, Kleinbock-Shi-Weiss, Björklund and Gorodnik: Haar measure satisfies condition (EMEI)

- We prove: Fix a Bernoulli measure μ on "nice" self-similar fractal in $Mat_{m\times n}$. Assume that there exists $\delta > 0$ and $k \geq 1$ such that for all $F \in C_c^\infty(\mathscr{X})$, and $x \in \mathscr{X}$, we have

$$\int_{\mathrm{M}_{m\times n}(\mathbb{R})} F(a_t u(\theta)x)\, d\mu(\theta) = \mu_{\mathscr{X}}(F) + O((x)^{-\beta} e^{-\delta t} \|F\|_{C^k}).$$

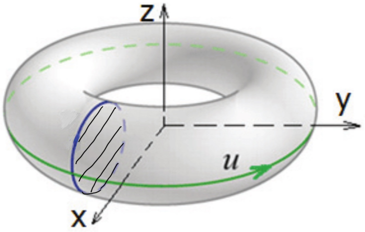
Then μ satisfies condition (EMEI).

Cross-Section

- Cross-section for flow $(a_t)_{t\in\mathbb{R}}$ on X is a subset S such that for every point of X, the set of visits times

$$\{t \in \mathbb{R} : a_t x \in S\}$$

is both discrete and totally unbounded.



Cross-Section

- **Return time function**

$$\tau(x) = \min\{t : t > 0, a_t x \in S\}.$$

- **First return map**

$$a_S(x) = a_{\tau(x)}(x).$$

- Kac formula

$$\int_S \tau(x)\, d\mu_S(x) = 1.$$

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Multiparameter flow

Need to study

$$\begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{-(t_1+t_2)} \end{pmatrix} \begin{pmatrix} 1 & & \theta_1 \\ & 1 & \theta_2 \\ & & 1 \end{pmatrix} \mathbb{Z}^3.$$

Problem

- No theory of cross-section for multiparameter flow exists!
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- Convert to counting problem
- Cross-section with almost correspondence with best approximation
  - Missed best approximation are negligible for result
- Almost surely, the number of hits to cross-section in increasing domain follow given asymptotic

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- Use Dani correspondence
- Construct cross-section for multi-parameter diagonal flow
- Genericity for almost every Λ_A under multi-parameter flow
- Use Schmidt counting result without constraints.

Idea of Proof

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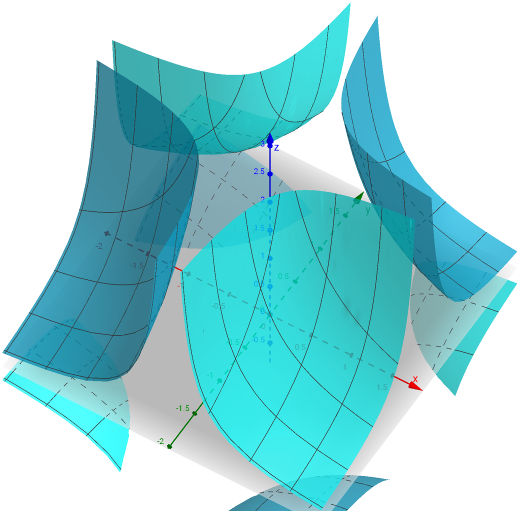
  - Approximants $\longleftrightarrow$ Points of lattice Λ_A in region



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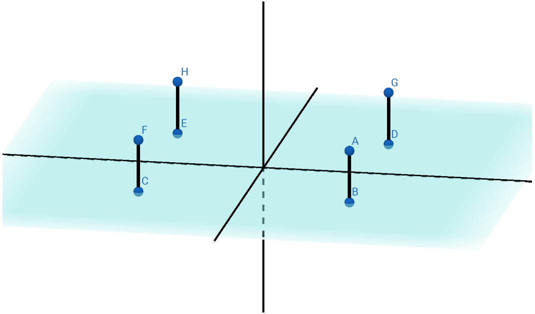

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Thank You!