Homogeneous Locally nilpotent derivations of rank 2 and 3 on $k[X, Y, Z]$

A joint work with Dr. Neena Gupta

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Contents

1. Preliminaries

Definition(Locally nilpotent derivation)

Let R be an integral domain containing a field k of charateristic zero . A function $D : R \to R$ is said to be a locally nilpotent derivation (LND) if it satisfies the following properties:

P1 $D(r+s) = D(r) + D(s)$ for all $r, s \in R$

P2 $D(rs) = rD(s) + sD(r)$ for all $r, s \in R$

P3 for every $r \in R$ there exists $n \in \mathbb{N}$ such that $D^n(r) = 0$

1. Preliminaries

Definition(Degree function)

Let G be a totally ordered abelian group. A function $\mu : R \to G \cup \{-\infty\}$ is said to be a degree function on R if it satisfies the following properties:

(a) $\mu(r) = -\infty$ if and only if $r = 0$.

(b) $\mu(rs) = \mu(r) + \mu(s)$ for every $r, s \in R$

(c) $\mu(r + s) \leq max\{\mu(r), \mu(s)\}$ for every $r, s \in R$

1. Preliminaries

Every locally nilpotent derivation D on R defines a degree function.

$$deg_D : R \to \mathbb{Z} \cup \{-\infty\}$$

for $r \neq 0$, $deg_D(r) := max\{n \in \mathbb{N} \cup \{0\} | D^n(r) \neq 0\}$
and $deg_D(0) = -\infty$

- $deg_D(r) = -\infty$ if and only if $r = 0$
- $deg_D(rs) = deg_D(r) + deg_D(s)$
- $deg_D(r + s) \leq max\{deg_D(r), deg_D(s)\}$

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Definition (Irreducible LND)

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Triangularizable LND

Let D be a locally nilpotent derivation on $k^{[n]}$. Then D is said to triangularizable if there exist a system of variables $X_1, ..., X_n$ such that

$$D(X_i) \in k[X_1, ..., X_{i-1}]$$

for $1 \leq i \leq n$

1. Preliminaries

Definition (Degree of a locally nilpotent derivation)

Let G be a totally ordered abelian group. $\mu : R \to G \cup \{-\infty\}$ be a degree function on R and $D \in LND(R)$. If maximum of the set $\{\mu(D(r)) - \mu(r) | r \in R, r \neq 0\}$ is in $G \cup \{-\infty\}$, then we define

$$degree_\mu(D) = max\{\mu(D(r)) - \mu(r) | r \in R, r \neq 0\}$$

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$$degree_{\mu}(D) = max\{\mu(D(r)) - \mu(r) | r \in R, r \neq 0\}$$

Definition(Homogeneous locally nilpotent derivation)

Let $R = \oplus_{n \in G} R_n$ be a G graded domain containing a field k.
$D \in LND(R)$ is said to be homogeneous with respect to the G grading if and only if there exists $d \in G$ such that $DR_n \subset R_{n+d}$ for all $n \in G$.

1. Preliminaries

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Definition (Rank of an LND)

Let D be a locally nilpotent R- derivation on the polynomial ring $R^{[n]}$. Then we define rank of D by:

$$min\{r \mid DV_1, ..., DV_r \neq 0 \text{ and } DV_{r+1} = ... = DV_n = 0$$
$$\text{where } V_1, V_2, ...V_n \text{ is a system of variables of } R^{[n]}\}$$

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

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Q. Does there exist a homogeneous locally nilpotent derivation on $k^{[n]}$ of degree 2 and rank n ?

We investigated the above question for $n = 3$.

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

- (Miyanishi, Kambayashi): For a field k of characteristic 0, if $D \in LND(k^{[3]})$,then $ker(D) = k^{[2]}$.

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- (D. Daigle): If $D \in LND(k^{[n]})$ and $ker(D) = k[F_1, ..., F_{n-1}]$, then $D = \alpha\Delta_{(F_1,...,F_{n-1})}$ for some $\alpha \in ker(D)$.

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- A homogeneous locally nilpotent derivation $D = \alpha\Delta_{(F,G)}$ on $k^{[3]}$ is said to be of type (l, m, n) if $deg(\alpha) = l$, $deg(F) = m$ and $deg(G) = n$.

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

Theorem 2.1

Let D be a homogeneous locally nilpotent derivation of $rank(D) > 1$ with respect to the standard grading $(1, 1, 1)$ on $k[X, Y, Z]$. Then there exist linear system of variables $\{L_1, L_2, L_3\}$ such that $deg_D(L_1) < deg_D(L_2) < deg_D(L_3)$

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- The above result generalizes for $R[X, Y, Z]$, where R is a PID.
- But it is not true for every one dimensional normal domain.

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

Theorem 2.2

With respect to the standard grading $(1, 1, 1)$ on $k[X, Y, Z]$, there is no homogeneous locally nilpotent derivation of type $(0, 3, 3)$ and $(0, 2, d + 1)$ for $d = 1, 2, 3$.

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

Let D be a homogeneous LND of degree d on $k[X, Y, Z]$ with respect to the standard grading $(1, 1, 1)$ and $D = \alpha\Delta_{(F,G)}$ for some $\alpha \in k[F, G]$. Then

$$d = deg(\alpha) + deg(F) + deg(G) - 3$$

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Let D be a homogeneous LND of degree d on $k[X, Y, Z]$ with respect to the standard grading $(1, 1, 1)$ and $D = \alpha\Delta_{(F,G)}$ for some $\alpha \in k[F, G]$. Then

$$d = deg(\alpha) + deg(F) + deg(G) - 3$$

- If $d = 0$, then either $deg(F) = 1$ or $deg(G) = 1$. So $rank(D) < 3$
- If $d = 1$ and $rank(D) = 3$, then D must be a homogeneous LND of type $(0, 2, 2)$.
- If $d = 2$ and $rank(D) = 3$, then D must be a homogeneous LND of type $(0, 2, 3)$.
- If $d = 3$ and $rank(D) = 3$, then D must be a homogeneous LND of type $(0, 2, 4)$ or $(0, 3, 3)$ or $(2, 2, 2)$.

2. Degrees of linear variables of $k[X, Y, Z]$ with respect to homogeneous LND and rank 3 derivations

Corollary 2.3

There is no homogeneous locally nilpotent derivation of rank 3 and degree ≤ 3 on $k[X, Y, Z]$ with respect to the standard grading $(1, 1, 1)$.

3. Homogeneous locally nilpotent derivation of rank 2 on $k[X, Y, Z]$

We investigated the structure of homogeneous locally nilpotent derivations of rank 2 on $k^{[3]}$ and characterised the triangularizable derivations among those.

3. Homogeneous locally nilpotent derivation of rank 2 on $k[X, Y, Z]$

Lemma 3.1

D be an irreducible homogeneous locally nilpotent derivation of rank 2 and degree d on $k[U, V, W]$ with respect to the standard grading. Then D is triangularizable if and only if there exists a system of variables $\{X, Y, Z\}$ linear in U, V, W such that $D = \Delta_{(X,P)}$ where

$$P = Y^{d+2} + Xf_{d+1}(X, Y) + \beta X^{d+1}Z$$

with $0 = deg_D(X) < deg_D(Y) < deg_D(Z), f_{d+1}(X, Y)$ is homogeneous polynomial of degree $d + 1$ and $\beta \in k^*$.
Moreover, $deg_D(Y) = 1$ and $deg_D(Z) = d + 2$.

3. Homogeneous locally nilpotent derivation of rank 2 on $k[X, Y, Z]$

Proposition 3.1

An irreducible homogeneous LND of rank 2 and degree $p-2$ on $k[U, V, W]$ is triangularizable, where p is a prime.

3. Homogeneous locally nilpotent derivation of rank 2 on $k[X, Y, Z]$

Proposition 3.1

An irreducible homogeneous LND of rank 2 and degree $p-2$ on $k[U, V, W]$ is triangularizable, where p is a prime.

It is clear from the Proposition 3.1 that the 2 is the smallest possible degree of an homogeneous LND of rank 2 which may not be triangularizable.

3. Homogeneous locally nilpotent derivation of rank 2 on $k[X, Y, Z]$

Theorem 3.1

Let D be an irreducible homogeneous locally nilpotent derivation of rank 2 and degree 2 with respect to the standard grading on $k[U, V, W]$ where k is algebraically closed. Then D is not triangularizable if and only if there exists a system of varible $\{X, Y, Z\}$ linear in U, V, W such that $D = \Delta_{(X,P)}$ where

$$P = (Y^2 + XZ)^2 + cX^3Y$$

for $c \in k^*$ with $0 = deg_D(X) < deg_D(Y) < deg_D(Z)$. Moreover, $deg_D(Y) = 2$ and $deg_D(Z) = 4$.

The Freeness Conjecture

Definition (Degree modules)

With respect to $D \in LND(R)$, the set $\mathscr{F}_n = \{r \in R |\ deg_D(r) \leq n\}$ is said to be the n-th degree A- module, where $A = ker(D)$.

The Freeness Conjecture

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Definition (D- basis)

Let $D \in LND(R)$ and $A = ker(D)$, a basis for a free A- submodule M of R is said to be a D- basis if every element of the basis has distinct deg_D value.

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Let $D \in LND(R)$ and $A = ker(D)$, a basis for a free A- submodule M of R is said to be a D- basis if every element of the basis has distinct deg_D value.

In [3] G. Freudenburg has conjectured the following:
Let $B = k^{[3]}$ and $D \in LND(B)$. If $A = ker(D)$, then B is a free A-module with basis $\beta = \{b_i|\ i \in \mathbb{N} \cup \{0\}\}$ where $deg_D(b_i) = i$.

4. Freeness of homogeneous triangularizable LND

Theorem 4.1

Let D be a triangularizable homogeneous locally nilpotent derivation on $B = k[U, V, W]$ and $A = ker(D)$. Then B is free A- module with a D- basis.

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Theorem 4.1

Let D be a triangularizable homogeneous locally nilpotent derivation on $B = k[U, V, W]$ and $A = ker(D)$. Then B is free A- module with a D- basis.

The homogeneous non-triangularizable locally nilpotent derivation of rank 2 and degree 2 has the freeness property.

Freeness Property

- For $D \in LND(k^{[n]})$, where $n \geq 4$, the freeness property does not hold.

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- The freeness property of any locally nilpotent derivation D is equivalent to the fact that

  For every n, the degree module $\mathscr{F}_n$ is a free A- module with a D-basis $\{b_i | \ 0 \leq i \leq n\}$.

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- D. Daigle has shown that every $\mathscr{F}_n$ is free A- module of rank $n + 1$.

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- D. Daigle has shown that every $\mathscr{F}_n$ is free A- module of rank $n + 1$.
- The existence of a D- basis is needed to be shown.

Example of a rank 3 R- derivation

Example

Let $B = R[X, Y, Z]$ where $R = \frac{\mathbb{R}[W_1, W_2]}{(W_1^2 + W_2^2 - 1)}$
w_1 and w_2 denote the residue classes of W_1 and W_2 in R respectively. For $d \geq 0$ we define a homogeneous locally nipotent R-derivation D of degree d on B as follows

$$DX = (1 - w_2)X_1^{d+1}$$

$$DY = -w_1 X_1^{d+1}$$

$$DZ = (d + 2)w_1 Y^{d+1}$$

where $X_1 = w_1 X + (1 - w_2)Y \in R[X, Y, Z]$

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Example of a rank 3 *R*- derivation

- $ker(D) \neq R^{[2]}$
- $rank(D) = 3$
- $deg_D(X) = deg_D(Y) = 1$ and $deg_D(Z) = d + 2$
- no linear system of variables $\{V_1, V_2, V_3\}$ such that

$$deg_D(V_1) < deg_D(V_2) < deg_D(V_3)$$

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