A short survey on numerical semigroups

Farewell Meeting for Dilip Patil

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An additively closed subset $S \subseteq \mathbb{Z}_{\geq 0}$ with $0 \in S$ is called a numerical semigroup if $\gcd(S) = 1$.

Let S be a numerical semigroup. Then there exist positive integers $a_1, a_2, \ldots, a_m$ with $\gcd(a_1, \ldots, a_m) = 1$ such that

$$S = \{c_1 a_1 + c_2 a_2 + \cdots + c_m a_m : \ c_i \in \mathbb{Z}_{\geq 0} \text{ for } i = 1, \ldots, m\}$$

We write

$$S = \langle a_1, \ldots, a_m \rangle.$$

The integers $a_1, \ldots, a_m$ are called the gnerators of S. Each numerical has a unique minimal set of generators.

The number of elements in a minimal set of generators is called the *embedding dimension* of S

There exists an integer $F(S)$ with the property that $F(S) \notin S$, but $a \in S$ for all $a > F(S)$. The integer $F(S)$ is called the *Frobenius number* of S.

Here is an example of a numerical semigroup

$$\cdots - -0 - - - -5 - 7 - -10 - 12 - 14, 15 - 17 - 19, 20, 21, 22 - 24 \cdots$$

It is the numerical semigroup $S = \langle 5, 7 \rangle$. Its Frobenius number is 23.

The elements of $\mathbb{Z}_{\geq 0}$ which do not belong to S are called the *gaps* of S and the elements of $a \in S$ with $a < F(S)$ are called the *non-gaps* of S.

The number of gaps is denoted by $g(S)$ and the number of non-gaps is denoted by $n(S)$.

Obviously one has

$$g(S) + n(S) = F(S) + 1.$$

If $a \in \mathbb{Z}_{\geq 0}$ is a gap, then $F(S) - a$ is a non-gap.

This implies that $n(S) \leq g(S)$.

In our example we have

$$a \in \mathbb{Z}_{\geq 0} \text{ is a gap if and only if } F(S) - a \text{ is a non-gap.}$$

Numerical semigroups with this property are called *symmetric*.

Any 2-generated numerical semigroup is symmetric. But there are more symmetric semigroups. For example, $S = \langle 4, 5, 6 \rangle$ is also symmetric.

There is an interesting algebraic interpretation for symmetric numerical semigroups.

Let K be a field. The toric K-algebra

$$K[S] = K[t^a : a \in S] \subseteq K[t].$$

is called the semigroup ring of the numerical semigroup S.

Here $K[t]$ is the polynomial ring in the variable t.

Obviously, $K[S]$ is a 1-dimensional Cohen-Macaulay domain whose integral closure is $K[t]$. The following result was shown by Kunz [4]

Theorem. S is symmetric if and only if $K[S]$ is Gorenstein.

Let $S = \langle a_1, \dots, a_m \rangle$. We consider the K-algebra homomorphism

$$R := K[x_1, \dots, x_n] \to K[S], \quad x_i \mapsto t^{a_i} \text{ for } i = 1, \dots, m$$

Then $K[S] \simeq R/I_S$. The ideal I_S is the *relation ideal* of S. It is a prime ideal of height $m-1$, generated by binomials.

What is known about I_S? Maybe the first result appeared in my dissertation [3] from 1969.

Let $S = \langle a_1, a_2, a_3 \rangle$. For $i = 1, 2, 3$ let c_i be the smallest positive integer such that $c_i a_i \in \langle a_k, a_l \rangle$ with $\{i, k, l\} = \{1, 2, 3\}$. Then for $i = 1, 2, 3$ there exist non-negative integers r_{ij} such that

$$c_i a_i = r_{il} a_l + r_{ik} a_k$$

Theorem. With the assumptions and notation introduced before one has $\mu(I_S) \leq 3$.

If $\mu(I_S) = 2$, then I_S is a complete intersection.

Otherwise, I_S is the ideal of 2-minors of the matrix

$$\begin{pmatrix} x_3^{r_{23}} & x_1^{r_{31}} & x_2^{r_{12}} \\ x_2^{r_{32}} & x_3^{r_{13}} & x_1^{r_{21}} \end{pmatrix}.$$

In 1976, Delorme [3] characterized all numerical semigroups for which I_S is a complete intersection.

Theorem. Let S be a numerical semigroup. Then S is a complete intersection if and only if S is obtained from $\mathbb{Z}_{\geq 0}$ by a sequence of iterating gluings.

In 1975, Bresinsky [1] showed that for each integer $m \geq 4$ and each integer $r > 0$ there exists a numerical semigroup S generated by m elements with $\mu(I_S) \geq r$. In contrast to this result, Bresinsky [2] proved in the same year

Theorem. If S is generated by 4 elements and S is symmetric, then $\mu(I_S) = 3$ or $\mu(I_S) = 5$.

Since height$(I_S) = 3$ and I_S is a Gorenstein ideal, I_S is generated by Pfaffians of a skew-symmetric matrix according to the Buchsbaum-Eisenbud structure theorem.

In 1993, Dilip [5] studied the case of a numerical semigroup S which is generated by a sequence of e positive integers where some $e-1$ of them form an arithmetic sequence.

Theorem. Let $p = e - 1$. then $\mu(I_S)$ is one of the numbers $p(p-1)/2 + p - r + 2$, $p(p-1)/2 + p - r' + 2$ and $p(p-1)/2 + 2p - r - d + 3$, where the integers r and r' result form the arithmetic of the sequence of generators and can be explicitly computed.

Moreover, Dilip gives an explicit description of the binomials which form a minimal set of generators of I_S.

In [1], Gimenez, Sengupta and Srinivasan identified Patil's ideal as a sum of two determinantal ideals, when the generators actually form an arithmetic sequence.

For $R = K[S]$, we now consider the canonical module of ω_R of R.

Since R is a graded Cohen-Macaulay domain, ω_R is graaded fractionary ideal.

More precisely, one has

Theorem. There is an isomorphism

$$\omega_R \simeq (t^{-c} : \ c \in \mathcal{G}(S))$$

of graded R-modules, where $\mathcal{G}(S)$ is the set of gaps of S.

The relative ideal semigroup ideal corresponding to ω_R is

$$\Omega_S = (-c : \; c \in \Gamma(S))$$

It is called the *canonical ideal* of S.

Let $M = S \setminus 0$ the maximal ideal of S. The elements of the set

$$PF(S) = \{a \in \mathbb{Z} \setminus S : \; a + M \in S\}$$

are called the *pseudo-Frobenius numbers* of S.

The cardinality of $PF(S)$ is the Cohen–Macaulay type $t(R)$ of $R = K[S]$.

Ω_S is minimally generated by the set $\{-c : \; c \in PF(S)\}$.

Let $(R, \mathfrak{m})$ be 1-dimensional local ring with canonical module ω_R. Then R is Gorenstein if and only if $\omega_R \simeq R$.

Goto, Takahashi and Taniguchi [2] developed the theory of almost Gorenstein rings.

R is *almost Gorenstein*, if there exists an exact sequence

$$0 \to R \to \omega_R \to C \to 0$$

with $\mathfrak{m}C = 0$.

The numerical semigroup S is called *almost Gorenstein*, if $K[S]$ is almost Gorentein.

The following characterization of almost Gorenstein semigroups is due to Nari [3].

Theorem. Let S be a numerical semigroup with

$$PF(S) = \{f_1 < f_2 < \ldots < f_{t(S)} = F(H)\}.$$

Then the following conditions are equivalent:

(i) S is almost symmetric.

(ii) $f_i + f_{t(S)-i} = F(S)$ for $i = 1, \ldots, t(S) - 1$

Almost Gorenstein 3-generated numerical semigroups can be characterized by the relation matrix of the defining ideal of its semigroup ring, as shown by Nari, Numata and K. Watanabe [4]

Let as before

$$\begin{pmatrix} x_3^{r_{23}} & x_1^{r_{31}} & x_2^{r_{12}} \\ x_2^{r_{32}} & x_3^{r_{13}} & x_1^{r_{21}} \end{pmatrix}$$

be the relation matrix of the 3-generated numerical semigroup.

Theorem. The following conditions are equivalent:

(i) S is almost symmetric.

(ii) The relation matrix is of the form

$$\begin{pmatrix} x_3 & x_1 & x_2 \\ x_2^{r_{32}} & x_3^{r_{13}} & x_1^{r_{31}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_3^{r_{23}} & x_1^{r_{21}} & x_2^{r_{12}} \\ x_2 & x_3 & x_1 \end{pmatrix}.$$

Let $(R, \mathfrak{m})$ by a local ring with canonical module ω_R, and let M be a finitely generated R-module.

One defines the trace of M, denoted $\text{tr}(M)$, as the sum of the ideals $\varphi(M)$, where the sum is taken over all R-module homomorphisms $\varphi\, M \to R$.

The trace of ω_R describes the non-Gorenstein locus of R.

In particular, R is Gorenstein if and only if $\text{tr}(\omega_R) = R$.

In [1], together with Hibi and Stamate, we called R *nearly Gorenstein*, if $\mathfrak{m} \subseteq \text{tr}(\omega_R)$. A numerical semigroup S is called nearly Gorenstein if $K[S]$ is nearly Gorenstein.

If a numerical semigroup is almost Gorenstein, then it is nearly Gorenstein.

A general classification of nearly Gorenstein rings seems to be impossible. But for 3-generated numerical semigroups with relation matrix

$$\begin{pmatrix} x_3^{r_{23}} & x_1^{r_{31}} & x_2^{r_{12}} \\ x_2^{r_{32}} & x_3^{r_{13}} & x_1^{r_{21}} \end{pmatrix}$$

we have the following result [2].

Theorem. Let $r_1 = \min\{r_{21}, r_{31}\}$, $r_2 = \min\{r_{12}, r_{32}\}$ and $r_3 = \min\{r_{23}, r_{13}\}$. Then S is nearly Gorenstein if and only if $r_1 = r_2 = r_3 = 1$.

Let $R = K[S]$. We denote by $\operatorname{res}(S)$ the length of $R/\operatorname{tr}(\omega_R)$.

$\operatorname{res}(S) \leq 1$ if and only S is a almost Gorenstein.

Theorem. Let S be a 3-generated numerical semigroup. Then with the notation introduced above we have

$$\operatorname{res}(S) = r_1 r_2 r_3.$$

and

$$g(S) - n(S) \leq \operatorname{res}(S).$$

Conjecture. The inequality $g(S) - n(S) \leq \operatorname{res}(S)$ is valid for any numerical semigroup.

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