IDEAL GENERATORS OF PROJECTIVE MONOMIAL CURVES IN $\mathbb{P}^3$. FAREWELL CELEBRATION FOR DILIP PATIL, JULY 29, 2021

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1. Introduction

This talk is based on [lpr] P. Li, D.P. Patil and L. Roberts, Bases and Ideal Generators for Projective Monomial Curves, Communications in Algebra, 40 (1), pages 173-191, 2012, which was my last paper with Dilip. First recall some notation from this paper.

We consider the ideal generators of projective monomial curves of degree d in $\mathbb{P}^3$.

Let $\mathscr{S} = \{a, b, d\}$ with $a, b, d \in \mathbb{N}$, $0 < a < b < d$ and $\gcd(a, b, d) = 1$. Let $S \subset \mathbb{N}^2$ be the semigroup generated by $\boldsymbol{\alpha}_0 = (d, 0), \boldsymbol{\alpha}_1 = (d-a, a), \boldsymbol{\alpha}_2 = (d-b, b), \boldsymbol{\alpha}_3 = (0, d)$.

Let K be a field and s, t indeterminates over K. Identify the semigroup ring of S over K with the subalgebra $K[S] = K[s^d, s^{d-a}t^a, s^{d-b}t^b, t^d] \subseteq K[s, t]$. Let $R = K[X_0, X_1, X_2, X_3]$, a polynomial ring over K. Let $\varphi : R \to K[s, t]$ be defined by $\varphi(X_0) = s^d, \varphi(X_1) = s^{d-a}t^a, \varphi(X_2) = s^{d-b}t^b, \varphi(X_3) = t^d$. Then φ is a surjective ring homomorphism, $\ker \varphi =: \mathfrak{p}$ is a homogeneous prime ideal in R, and $R/\mathfrak{p} = K[S]$. To $\mathscr{S}$ we associate the degree d "physical" curve $\mathcal{C} = \mathrm{Proj}(K[S]) \subset \mathbb{P}^3$.

The ideal $\mathfrak{p}$ has a minimal set $\mathscr{G}$ of pure binomial generators. In [lpr] we defined $f \in \mathscr{G}$ to be a type one generator if f does not have X_0 in one term and X_3 in the other, and a type two generator otherwise. Thus a type one generator contains one term of the form $X_1^{a_1}X_2^{a_2}$ where $a_1 \geq 0, a_2 \geq 0, a_1 + a_2 > 0$. The type two generators, by homogeneity considerations must be of the form $X_0^{a_0}X_2^{a_2} - X_1^{a_1}X_3^{a_3}$ with $a_0, a_1, a_2, a_3 > 0$. Now consider $K[S]/(s^d, t^d)K[S] \cong K[X_0, ..., X_3]/(\mathfrak{p}, X_0, X_3) \cong K[X_1, X_2]/J_1$ for a monomial ideal J_1 in $K[X_1, X_2]$. This yields a bijection between the type one generators of $\mathfrak{p}$ and the minimal monomial generators of J_1. To capture the type two generators we factor out by (X_1, X_3). Thus $K[S]/(s^d, t^d)K[S] \cong K[X_0, ..., X_3]/(\mathfrak{p}, X_1, X_3) \cong K[X_1, X_3]/J_2$ for a monomial ideal J_2 in $K[X_0, X_2]$. For a suitable $\mathscr{G}$ this yields a bijection between the type two generators of $\mathfrak{p}$ and *some* of the minimal monomial generators of J_2. It turns out that the (exponent vectors of) the generators of J_1 and J_2 (in the respective planes (X_1X_2) and (X_0X_2)) lie in the Hilbert basis of lattice points in a cone and hence lie on line segments which can be calculated using continued fractions. We get a collection of vertices and a finite number of integer points between them. This permits very efficient calculation of the ideal generators of $\mathfrak{p}$. Also the generators can be described by listing the endpoints of the intervals and the number of subdivisions.

Example 1.1. For $\mathscr{S} = \{1, 6, 7\}$ we have a segment of type two generators $\{X_0X_2^5 - X_1^2X_3^4, X_0^2X_2^4 - X_1^3X_3^3, X_0^3X_2^3 - X_1^4X_3^2, X_0^4X_2^2 - X_1^5X_3\}$. This is a "segment" because the exponent vectors of the X_0-X_2 terms, namely $\{(1, 5), (2, 4)(3, 3), (4, 2)\}$ lie on a straight line. Instead of listing all four generators one can just give the endpoints $\{(1, 5), (4, 2)\}$ and the number of subintervals this gets divided into, namely 3. There are three type one generators $\{X_0X_3 - X_1X_2, X_1^6 - X_0^5X_2, X_2^6 - X_3^5X_1\}$. More generally $\mathscr{S} = \{1, n-1, n\}$ has three type one generators, and a segment of $n-3$ type two generators, yielding n generators in total. This is the largest possible number of generators for a projective monomial curve of degree n in $\mathbb{P}^3$. If n is larger, say 1000, we will get a segment containing 997 type two generators, and we really do not want to see all of them. The end points and number of subintervals suffices to describe the generators.

2. More technical properties.

We define a projective monomial curve to be Cohen-Macaulay if R is Cohen-Macaulay. There are many characterizations of Cohen-Macaulay. In our case perhaps the simplest is to say that R is Cohen-Macaulay if and only if s^d, t^d is a regular sequence.

(1) A complete intersection is Cohen-Macaulay. (2 ideal generators)
(2) A non-obvious example we worked out explicitly is $\mathscr{S} = \{4, 7, 13\}$.
(3) A projective monomial curve in $\mathbb{P}^3$ is Cohen-Macaulay if and only if it has no type two generators.
(4) A projective monomial curve in $\mathbb{P}^3$ is Cohen-Macaulay if and only if it has less than or equal to 3 generators.

3. More Speculative

(1) As the degree goes to infinity the average number of segments of ideals remains small, maybe less than one, depending on how segments are counted.
(2) Item (1) notwitstanding, the number of type one or type two ideal generator segments can be made arbitrarily large.
(3) As the degree becomes large the fraction of curves that are Cohen-Macaulay approaches about .31.
(4) More generally the fraction of curves with a specified number of ideal generators approaches a value approximated in the following table

Table 1

d	2	3	4	5	6	7	8	9	10	11	12
10^7	0.0012	0.3034	0.1176	0.1061	0.0884	0.0700	0.0540	0.0412	0.0316	0.0245	0.0194
10^{10}	0	0.3016	0.1190	0.1015	0.0890	0.0706	0.0562	0.0415	0.0329	0.0243	0.0186
10^{21}	0	0.3072	0.1170	0.1050	0.0864	0.0676	0.0545	0.0401	0.0323	0.0226	0.0199
10^{41}	0	0.3102	0.1154	0.1085	0.0869	0.0687	0.0516	0.0398	0.0295	0.0252	0.0194
standard sample	0	0.3064	0.1171	0.1050	0.0874	0.0690	0.0541	0.0404	0.0316	0.0240	0.0193

The "standard sample" is obtained from about 60000 curves of degrees $10^{10}, 10^{21}, 10^{41}$ computed once and for all, hence is a sort of average of the prevous three rows of the table. This of course proves nothing, but the values are remarkably stable.

(5) The median number of ideal generators is 5 if the degree is sufficently large. This is suggested by the sums $0.3102 + 0.1154 = .4256 < 5$ and $0.3102 + 0.1154 + 0.1085 = .5341 > .5$.
(6) For no degree is the median number of generators greater than 5. This was tested up to a few hundred.
(7) The preceeding items (1), (2) (5) (6) not withstanding, the average number of ideal generators goes to infinity as the degree becomes large.

4. Who cares?

Maybe no one. If one can compute up to degree 100 or so, one might feel that one has a pretty good program. However one would be mislead in several ways, thinking for example that there are more complete intersections or that there can be only one segment with a large number of ideal generators. We did not find much new happening after degree about 10^7, but one never knows for sure. The highest degree of a monomial curve that we tested was about 10^{10000}.

Also knowing the complexity of a computation is always of interest. However defined, one would expect the complexity of the computation to be at least as large as the answer. But here, as one is allowed to describe the ideal generators using segments, we can do better, and the complexity of the ideal generation computation for projective monomial curves of degree d in $\mathbb{P}^3$ seems to be a small power (maybe 3) of $\log d$. However we felt on rather shaky ground here.

5. Forgotten comments

Projective monomial curves can be defined in $\mathbb{P}^n$ for any $n \geq 2$. If $n = 2$ all curves are Cohen-Macaulay. If $n \geq 4$ the ideals are still generated by pure binomials but one loses the continued fraction method of calculation. Les Reid and I showed that the fraction of all projective monomial curves of degree n that are Cohen-Macaulay approaches 0 as n goes to infinity. I forgot to make these comments during my talk.

A manuscript giving more details of this talk and references will be posted on the Queen's web page of Ping Li.

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