Correlations in multispecies asymmetric exclusion processes

A Dissertation
submitted in partial fulfilment
of the requirements for the award of the
degree of
Doctor of Philosophy

by
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Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Arvind Ayyer at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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10-06-00-10-31-14-1-11381

Indian Institute of Science,
Bangalore,
April, 2023.

Prof. Arvind Ayyer
(Research advisor)

Dedicated to

My brother,
Harshit Pahuja.

Acknowledgments

To begin, I would like to convey my heartfelt appreciation to my research supervisor Prof. Arvind Ayyer who has guided me through my graduate studies. His mentorship, guidance, and constant support have been instrumental in shaping me into a better researcher and individual. I am grateful for his patience and understanding. I extend my sincere thanks to him for providing me with the opportunity to visit some of the best conferences and workshops. I am deeply grateful to him for carefully reviewing all the drafts of my research articles, as well as this thesis, and for providing me with valuable feedback that significantly improved the quality of my work. I am highly appreciative of his attention to detail and his commitment to helping me succeed.

I would like to extend my sincere thanks and gratefulness to the faculty members of the Department of Mathematics at IISc for offering a wide variety of courses which were extremely beneficial to me. I am thankful for the opportunity to be a teaching assistant to Prof. Vamsi Pingali, Prof. Soumya Das, Prof. Thirupathi Gudi and Prof. Sanchayan Sen. It was not only a teaching experience but also a great learning experience for me. I would like to extend a special thanks to Prof. Manjunath Krishnapur for his accessibility and willingness to listen to my concerns and provide me with the necessary advice and direction. I would like to express my humble gratitude and sincere thanks to Prof. Amritanshu Prasad and my advisor for arranging a research visit to IMSC Chennai, where I had the privilege of interacting with and learning from Prof. Xavier Viennot. I also extend my heartfelt appreciation to Prof. Viennot for his eagerness to share his knowledge and for engaging in insightful discussions. I express my special thanks to Prof. Svante Linusson for a valuable suggestion which helped me improve a result of Chapter 3. I am thankful to my collaborator Surjadipta De Sarkar who has been extremely supportive and fun to work with. I express my gratitude to my colleagues Subhajit Ghosh, Nishu Kumari, G.V.K. Teja, and Anita Arora for insightful discussions and fruitful learnings.

It is my pleasure to have a wonderful class of colleagues/friends at the Department of Mathematics, IISc. I want to express my thanks to all of them. My special thanks to Aakanksha Jain, Abhay Jindal, Abu Sufian, Annapurna Banik, Poornendu Kumar, Prateek Kumar Vishwakarma, Pramath Anamby, Purba Banerjee, Ramesh Chandra Sau, Sahil Gehlawat, Sanjay Jhawar, Shubham Rastogi, and all my Integrated PhD batch-mates for many helpful academic and non-academic discussions. I would like to thank the staff at the Mathematics department for their hard work and dedication, which created a welcoming and supportive environment for students. Their contributions, from administrative support to technical assistance, have made my academic journey smoother and more enjoyable. I would like to acknowledge the IISc integrated PhD scholarship and the support in part from a SERB grant CRG/2021/001592.

My friends have been an integral part of my life at IISc, and they have provided me with the necessary motivation and inspiration to succeed in my academic pursuits. I convey my profound thanks to my friends Rahul Biswas, Sruthi Sekar, Swati Gupta and Anupam Sanghi for their presence in my life and for being there for me through thick and thin. Last but not least I would also like to thank my friends Maitreyee Rudola, Moureena Khokhar, Sheetal, Mansimram Singh, Swati Agarwal, Payal Sharma, Hitesh Basantani, Nikhil Gulati, Shivangi Singhal and Ashutosh Kumar for always believing in me and encouraging me.

My teachers have been instrumental in shaping me into a better individual and researcher. I take this opportunity to express my gratitude to all my teachers. I extend my sincere thanks to Nandita Narain, Archana Chopra, Kashif Ahmed, Maria Thomas and Sonia Dawar, all the teachers at St. Stephens College, Mathematics Department during 2011-2014, for their role in my undergraduate student life. Their passion for teaching and dedication to my success are deeply appreciated.

Finally, I wish to express my indebtedness to my parents, my grandmother and my sister-in-law. They have made it possible for me to pursue my dreams and achieve my academic goals. I am grateful for their unconditional love, guidance, and sacrifices. I would like to express my special thanks to my brother Harshit Pahuja, who instilled in me a love for Mathematics and has been my pillar of support throughout my academic journey. His constant encouragement, humour, and support have kept me motivated during challenging times. I am grateful for his unwavering belief in me and his constant push to be my best.

Abstract

The main aim of this thesis is to study the correlations in multispecies exclusion processes inspired by the research of Ayyer and Linusson (Trans. AMS., 2017) where they studied correlations in the multispecies TASEP on a ring with one particle of each species. The focus is on studying various models, such as *multispecies TASEP on a continuous ring*, *multispecies PASEP on a ring*, *multispecies B-TASEP* and *multispecies TASEP on a ring with multiple copies of each particle*. The primary goal is to investigate the two-point correlations of adjacent particles in these models. The details of these models are given below:

We study the multispecies TASEP on a continuous ring and prove a conjecture by Aas and Linusson (AIHPD, 2018) regarding the two-point correlations of adjacent particles. We use the theory of multiline queues developed by Ferrari and Martin (Ann. Probab., 2007) to interpret the conjecture in terms of the placements of numbers in triangular arrays. Additionally, we use projections to calculate correlations in the continuous multispecies TASEP using a distribution on these placements.

Next, we study the correlations of adjacent particles on the first two sites in the multispecies PASEP on a finite ring. To prove the results, we use the multiline process defined by Martin (Electron. J. Probab., 2020), which is a generalisation of the multiline process defined earlier by Ferrari and Martin.

We then study the multispecies B-TASEP with open boundaries. Aas, Ayyer, Linusson and Potka (J. Physics A, 2019) conjectured a formula for the correlations between adjacent particles on the last two sites in the multispecies B-TASEP. To approach this problem, we use a Markov chain that is a 3-species TASEP defined on the Weyl group of type B. This allows us to make conjectures and prove some results towards the above conjecture.

Finally, we study a more general multispecies TASEP with multiple particles for each species. We extend the results of Ayyer and Linusson to this case and prove formulas for two-point correlations and the TASEP speed process.

Contents

Chapter 1

Introduction

This thesis focuses on the study of two-point correlations in different types of multispecies asymmetric simple exclusion processes (ASEPs) using probabilistic and combinatorial methods. The exclusion process is a simple yet fundamental model that describes the probabilistic movement of particles on a lattice, where each site can be occupied by at most one particle. It is a Markov process where the transitions are defined by letting the particles jump or swap randomly according to certain stochastic rules. In an ASEP, the rates at which they move to the left and right are different. When particles are biased to move only in one direction, we refer to the model as a totally asymmetric simple exclusion process or a TASEP, due to the extreme asymmetry. Exclusion processes are investigated in many different settings, and many of their properties are of interest to probabilists, combinatorialists and statistical physicists. One such property is the correlation of two or more particles in the stationary distribution of a process. This line of study was initiated by Ayyer and Linusson [20] where they studied correlations in the multispecies TASEP on a ring in order to prove conjectures by Lam [71] on random reduced words in an affine Weyl group. In particular, they gave results on the correlations of two adjacent points in the multispecies TASEP on a ring. We extend their results in a few different directions in the next few chapters.

Exclusion processes are an example of a commonly studied class of Markov chains known as *Interacting Particle Systems* [48, 75, 76]. In a seminal work in 1970, Frank Spitzer [94] introduced many different models of interacting particle systems namely the exclusion process, zero-range process, contact process, voter model, long-range exclusion process and so on. We work with the exclusion processes in one dimension. We begin by surveying related works in this chapter while discussing some of the applications of ASEPs in other areas. Later, we give an overview of the organisation and the layout of the thesis.

1.1 Brief Literature Review

This section highlights key findings in the study of exclusion processes. The *Asymmetric Simple Exclusion Process* (ASEP) is a fundamental model that simulates the movement of particles on a lattice, and it has been widely explored in many variations. The earliest known publication of the ASEP was by MacDonald, Gibbs, and Pipkin in 1970 as a prototype to model the dynamics of ribosomes along RNA [77]. The model's simplicity and generality make it a valuable tool for understanding the behaviour of a wide range of systems, while its rich macroscopic behaviour makes it an interesting subject of study in its own right. Many techniques to solve the exclusion process have been developed and exact results have been derived since its introduction. The term *Exclusion Process* was coined by Spitzer [94] and the word "exclusion" refers to the constraint that there can be at most one particle at each site of the lattice. The sites that are empty are also referred to as *holes*. An exclusion process is *simple* if each jump of any particle is to an adjacent site on the lattice. If the particles prefer to jump in a particular direction, the model is known as an asymmetric exclusion process. In other words, in an ASEP, particles can carry out forward exchanges with holes at a rate of p and backward exchanges at a rate of q where $0 \leq q < p \leq 1$. Depending on the value of q, an ASEP can be of two kinds; *totally asymmetric exclusion process* or TASEP where $q = 0$ and p is usually scaled to 1 and *partially asymmetric exclusion process* or PASEP where $0 < q < p$. An exclusion process is called a symmetric simple exclusion process if $p = q$.

Depending on the requirements of the model, the asymmetric simple exclusion process can be subjected to different boundary conditions. The two main boundary conditions are *open boundaries* and *periodic boundaries*. In an open ASEP, particles can enter and exit the system at the ends of the lattice. From a physicist's point of view, models that are defined on lattice paths with open boundaries connected to reservoirs, that allow for the exchange of particles, are considered to be more realistic. A common illustration of such a system is a pipe connecting two or more reservoirs at different temperatures or chemical potentials. The exact solution of the single species TASEP with open boundary conditions was determined in 1993 by Derrida, Evans, Hakim and Pasquier [40] using a technique called *matrix product ansatz*. Independently, Domany and Schutz [91] derived the exact phase diagram of the same model using recursion relations on the size of the system. The seminal contributions of the discovery of boundary-induced phase transitions in the open exclusion process by Ferrari and Krug [66] and later, of the matrix ansatz solution [40] have significantly advanced the field. The matrix product formulation of the steady state on the open boundary system for the single species PASEP has also been fully worked out [25, 87]. Appropriate representations relevant to the matrix product ansatz for these variants were also studied in [49, 86]. Other notable contributions towards the single-species exclusion process with open boundaries can be found in various references; see [38, 99].

For the periodic boundary variation of the ASEP, the particles move on a lattice of finite size and the number of particles is conserved. A single species ASEP on a periodic boundary has a uniform stationary distribution. The 2-species asymmetric exclusion process is a generalisation of the single-species ASEP where there are two kinds of particles occupying the sites of the lattice such that there is at most one particle at each site. Much of the early investigation towards the solution of the 2-species TASEP was done with the motivation to study the shock measures in various particle systems [51, 53, 55, 93]. The steady-state distribution of the two-species exclusion process on a *ring* $\mathbb{Z}/n\mathbb{Z}$, also known as exclusion process with *periodic boundaries*, was obtained by Derrida, Janowsky, Lebowitz and Speer [43] using the matrix product representation introduced in [40]. The exact solution was generalised to different models of the ASEP namely with *impurity* [78] and to *infinite systems* [44]. The 3-species TASEP on a ring was also studied and solved exactly by Mallick, Mallick and Rajewsky [79] using a matrix ansatz. Combinatorially, the stationary distribution for the 2-type TASEP was described around the same time both by Angel [8] and by Duchi and Schaeffer [47]. Ferrari and Martin [56, 57] improved these by constructing *multiline queues* as a device to study the TASEP with *multiple* species of particles by using ideas from Ferrari, Fontes and Kohayakawa [54]. The construction of Ferrari and Martin inspired the later works where their algorithm was transformed into a matrix product representation of the multispecies TASEP [11, 50]. In 2009, the exact solution for the stationary state measure for the PASEP on a ring with multiple species was found by Prolhac, Evans and Mallick [83]. This was achieved by extending the matrix-product representation from [50] to $q > 0$. For periodic boundary conditions, the matrix ansatz was developed for the multispecies PASEP in [12] as well. Using the matrix representation, the probabilistic construction of Ferrari and Martin [57] was also generalised to the multispecies PASEP by Martin [80] by providing a recursive construction of the stationary distribution. Ferrari and Martin's multiline queues have been widely utilised by researchers to demonstrate nice properties in many different models of exclusion processes. The recursive approach to the construction of multispecies particle systems has been extended to a range of different particle systems, including discrete-time TASEPs [11, 81], inhomogeneous versions of the multi-type TASEP [15, 19, 27], and a variety of zero-range processes [21, 22, 68, 69, 70]. Many different properties of the multispecies TASEP on a ring have been explored further; see [4, 5, 20].

Among the open boundaries, the ASEP with *semipermeable boundaries* is an interesting class on its own. Here, different passage rules are applied to different particle types, and no species is allowed to pass through both boundaries. The boundaries in the semipermeable ASEP exhibit a property known as integrability, which is useful for deriving the matrix product ansatz. The exact solution was derived by Arita [9, 10] in 2006 for the case of 2-species totally asymmetric exclusion process with semipermeable boundaries. Later, Ayyer, Lebowitz and Speer [17, 18] explained additional properties of the phase diagram of this model. The phase diagram for a more general version of the semipermeable ASEP was derived with the help of matrix ansatz and the appropriate representation in terms of orthogonal polynomials [98] and in terms of Koornwinder polynomials [28]. The two-species TASEP with open integrable conditions was discussed and solved in [34, 35]. The multispecies version of the semipermeable model was further investigated in [23, 30, 58]. Recent publications by Aas, Ayyer, Linusson and Potka [1, 2] also study a variant of the multispecies TASEP on finite sites with open boundaries.

Lam [71] studied Markov chains that can be represented geometrically as random walks on a regular tessellation of a vector space with the action of an affine Weyl group. In the case of a symmetric group S_n, Lam and Williams [72] showed that the Markov chain can be interpreted as a process on permutations of n elements. Ayyer and Linusson [19] demonstrated that the Markov chain studied by Lam and Williams is equivalent to a multispecies TASEP with inhomogeneous transition rates. Arita and Mallick [15] furthered their work by constructing a matrix product solution of the system. These developments highlight that techniques developed in non-equilibrium statistical mechanics to study the ASEP can be applied to combinatorial problems and vice-versa. The connections of exclusion processes to families of symmetric polynomials such as Schubert polynomials and Macdonald polynomials have been explored by various authors in works like [27, 29, 32, 33, 64] using objects related to the multi-line diagrams. The ASEP speed process is studied in [6, 7] by considering an infinite volume limit of the multispecies ASEP.

ASEPs have been studied from several other perspectives, and the different physical properties of an exclusion process have been investigated. Some of the properties that are of interest are diffusion constants [36, 42], current fluctuations [39, 52, 84, 97], integrability [26, 59, 58], correlation functions [4, 7, 20, 41, 100], phase diagrams [1, 10], certain large deviation functions [37, 46, 45] and mixing times [61] on different models. Spectral properties of a few models are discussed in [3, 14, 27].

1.2 Applications of exclusion processes

The ASEP model is able to capture the essential features of a wide range of physical and biological systems, such as the transport of particles in a crowded environment, and the flow of traffic on a highway. The asymmetric simple exclusion process has been widely used in various fields of research, including physics, biology, and computer science.

1. The ASEP is a widely studied model in the field of physics, particularly in the context of statistical mechanics and condensed matter physics. It finds application in the study of driven diffusive systems [67] and spin chains [73]. Fast ionic conductors were modelled as stochastic lattice gas with excluded volume interaction in [63].

2. The ASEP is a useful model for studying transport flow in various systems. The multispecies version allows for different treatment of different classes of vehicles on a single-land road [62]. Other biological and physical transport phenomena like ant trails, pedestrian dynamics and intracellular transport can also be modelled using exclusion processes [31, 88, 101].

3. The ASEP has proven to be a valuable tool for understanding the dynamics of many biological systems including the movement of ribosomes along an RNA molecule [77], movement of motor proteins along a microtubule (also called *molecular motors*) [65], DNA replication [90], cellular automata [13, 89] and transport of neurotransmitters [92, 96].

4. ASEPs are used in other branches of mathematics. Queueing systems can also be modelled using the ASEP [16, 82]. The exclusion process can be used to study some variations of card shuffling models [24]. The growth process of *n-core partition*, a special class of integer partitions has also been understood in connection to the multispecies TASEP [20, 71].

1.3 Organisation of the thesis

We are motivated by the work of Ayyer and Linusson [20] to study different models of exclusion processes and find correlations of adjacent particles. In this thesis, we study different models of multispecies asymmetric exclusion process, namely *multispecies TASEP on a continuous ring*, *multispecies PASEP on a ring*, *multispecies B-TASEP* and *multispecies TASEP on a ring with multiple copies of each particle*. Our goal is to study the two-point correlations of adjacent particles in these models.

We use the theory of multiline processes to study the distribution of exclusion processes. This is done by utilizing a procedure known as *lumping* which projects a larger Markov process to a smaller Markov process, the latter being an exclusion process for our case. The organisation of the thesis is as follows:

In Chapter 2, we discuss the background material required for this thesis. We first discuss the probabilistic prerequisites including the basic theory of Markov processes and stationary distributions. We briefly describe the lumping process and explain the exclusion process in detail. This is followed by discussing the algebraic combinatorics techniques that we use in our proofs.

In Chapter 3, we prove a conjecture by Aas and Linusson [4] on the two-point correlations of adjacent particles in a continuous multispecies TASEP on a ring. We use the theory of multiline queues as devised by Ferrari and Martin [56, 57] to interpret the conjecture in terms of placements of numbers in triangular arrays. Further, we use projections to calculate correlations in the continuous multispecies TASEP using a distribution on these placements.

Following the combinatorial analysis of the multiline queue construction for the TASEP [20], it is natural to explore whether an analogous application of appropriate multiline queues could lead to similar results for multi-point probabilities for the partially asymmetric case. In Chapter 4, we solve this problem of correlations of adjacent particles on the first two sites in the multispecies PASEP on a finite ring. We use the multiline processes defined by Martin [80], the dynamics of which also depend on the asymmetry parameter q, to compute the correlations.

In Chapter 5, we study the correlations of adjacent particles on the last two sites in the multispecies B-TASEP. We use another Markov chain which is a 3-species TASEP defined on the Weyl group of type B, similar to the 2-species B-TASEP studied in [2] to give some results and conjectures towards the aforementioned correlation functions.

Ayyer and Linusson [20] studied the correlations of two or more particles in the multispecies TASEP on a ring with finite sites. In particular, they studied the correlations of the first two sites on the ring which has exactly one particle of each type. Finally, in Chapter 6, we extend their result to the multispecies TASEP such that there is an arbitrary number of particles of each type.

Chapter 2

Preliminaries

In this chapter, we lay out the basics required for the thesis. In Section 2.1, we review the basic concepts of Markov processes on finite state spaces and introduce the stationary distribution and correlation function for these processes. We also discuss some classic methods used in probability theory. Section 2.2 is dedicated to the algebraic combinatorics methods we use to prove our results.

2.1 Probability Theory Background

In this section, we discuss Markov processes with finite state space. We state a few results which will be helpful in the later chapters. Most of the notation in this section is borrowed from [74] and [85].

2.1.1 Markov processes

Consider a stochastic process $\{X_t, t \geq 0\}$ where X_t takes values from a set Ω of non-negative integers. Ω is called the *state space* of the process. If $X_t = i$, for some t then the process is said to be in *state* i at time t.

Definition 2.1. Let Ω be a finite set.The process $\{X_t, t \geq 0\}$ is said to be a *continuous-time Markov process* with state space Ω if for all $s, t \geq 0$ and $i, j \in \Omega$,

$$\mathbb{P}\{X_{(t+s)} = j | X_s = i, X(u) = x(u), 0 \leq u \leq s\} = \mathbb{P}\{X_{t+s} = j | X_s = i\}, \tag{2.1.1}$$

where $x(u)$ are non-negative integers for $u \in [0, s]$.

Equation (2.1.1), also known as the *Markov property*, means that a future state is dependent on the present state only and not on any past state. The quantity in (2.1.1) is denoted by $P_t(i,j)$. For all $i,j \in \Omega$, let q_{ij} denote the rate of transition of the process from j to i. The *transition rate matrix* $Q = (Q)_{i,j}$ is defined as

$$(Q)_{i,j} = \begin{cases} q_{ij}, & i \neq j, \\ -\sum_{k\neq i} q_{ki}, & i = j. \end{cases}$$

The Markov process $X(t)$ is said to be *irreducible* if

$$P_t(i,j) > 0, \text{ for all } i,j \in \Omega \text{ and } t > 0.$$

Let $\Pi = (\Pi_i; i \in \Omega)$ denote the long-term probability of being in state i, where $\Pi_i \geq 0$ for all i and $\sum_i \Pi_i = 1$. The vector Π, provided it exists, is also known as the *stationary distribution* of the process, and it satisfies the equation $\Pi Q = 0$. An irreducible Markov process has a unique stationary distribution.

2.1.2 Projection of chains

In probability theory, it is fairly common to create a new Markov process from an existing one through a process called *projection* or *lumping of chains.* Consider a Markov process $\mathbf{X} = \{X_t, t \geq 0\}$ with state space Ω and transition matrix $\mathbf{P}$ such that there is an equivalence relation $\sim$ that partitions Ω into equivalence classes. Let $[x]$ denote the equivalence class of $x \in \Omega$ such that $\mathbf{P}(x,[y]) = \mathbf{P}(x',[y])$ for $x \sim x', y \in \Omega$. Then, $[\mathbf{X}]$ is a Markov process with $\tilde{\Omega} = \{[x] : x \in \Omega\}$ as its state space (see [74, Lemma 2.5]). The transition matrix for the new Markov process is given by $\tilde{\mathbf{P}}$ where $\tilde{\mathbf{P}}([x],[y]) := \mathbf{P}(x,[y])$. We use this technique extensively in the following chapters of this thesis to prove our results.

2.1.3 Exclusion Process

We focus on the study of the one-dimensional asymmetric exclusion process. This process can be modelled on a path graph or a cycle graph with the constraint that there is at most one particle at each site. The empty sits are called *holes.* The particles move according to certain rules, such that they can hop to a neighbouring empty site but not occupy the same site. This is illustrated in Figure 2.1 for a path and a cycle graph. The rates at which these transitions occur are given by p, when the particle moves left (or clockwise) and q, when the particle moves rightwards (or counterclockwise).

In a multispecies exclusion process, the particles have a certain hierarchy, characterized by an integer labelling each particle. This label is known as the *type* or the

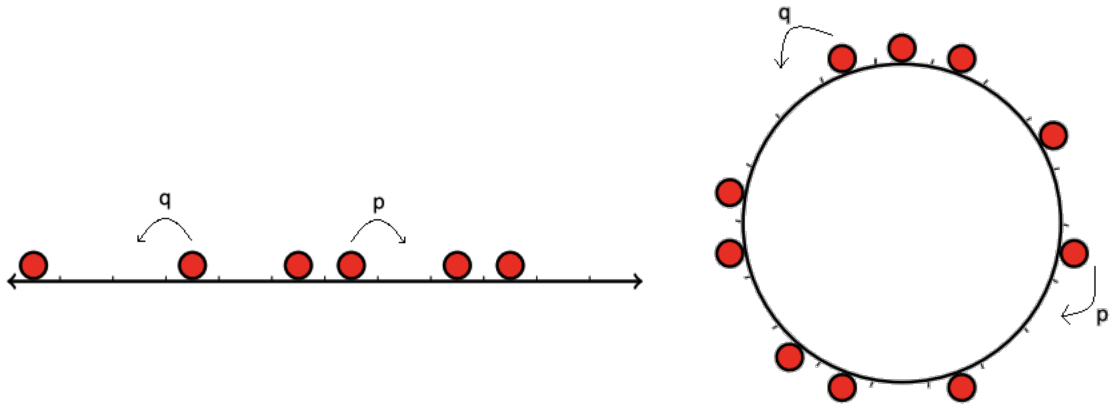


Figure 2.1: Transitions in an asymmetric exclusion process.

species of the particle. The ASEP with particles of types $\{1, \ldots, N\}$ (along with empty sites or holes) can be viewed as a coupling of N one-type ASEPs. Specifically, for any $n = 1, \ldots, N$, we can consider a projection under which types $r \leq n$ are considered "particles" and types $r > n$ are considered "holes" or "vacancies". For each n, the image of the process under this projection is a one-type ASEP. The state of the system at any given time is defined by the configuration of labelled particles and vacancies on the sites of the lattice. The vacant sites can be treated as particles with the highest label. Each particle carries an exponential clock which rings with rate 1, and the particle tries to jump to its neighbouring site whenever the clock rings. Let the former site be occupied with a particle labelled i and the latter be with a particle labelled j. The interactions happen with rates given as

$$ij \to ji \text{ with rate } \begin{cases} p, & \text{if } i > j, \\ q, & \text{if } i < j. \end{cases}$$

The exclusion process where the particles jump only in a preferred direction (towards the right in this case) with the rate 1 is known as *totally asymmetric exclusion process* or TASEP in short. That is, the TASEP is a special case of the ASEP with $p = 1$ and $q = 0$. For the multispecies case, if the site towards the right of a higher particle is occupied by a particle of lower species, the two particles exchange places with rate 1 (note that the holes are treated as particles of the highest type). The TASEP is known to be irreducible and therefore has a unique stationary distribution.

On the other hand, in a *partially asymmetric exclusion process* or PASEP, particles can jump both to the right (with rate p scaled to 1) and to the left (with rate $q \in (0, 1)$). In the multiple species case, a particle labelled i switches position with a particle labelled

j on its right with rate 1 if $i > j$, and with rate q if $i < j$. The rate q is called the *asymmetry parameter* of the PASEP.

A *multiline process* is also a Markov process defined on a structure which is a collection of disjoint path graphs or cycle graphs of the same length. It can also be seen as a stack of these graphs with particles occupying some of the sites on the graph. These structures are known as *multiline queues* (or MLQs). The multiline process can be projected to the multispecies exclusion process by lumping.

2.2 Algebraic Combinatorics Background

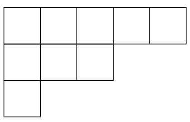
In this section, we see some of the key concepts from combinatorics that are useful in our thesis. We discuss topics like Young tableaux and their enumeration. Most of our notation is borrowed from [95].

2.2.1 Standard Young tableaux

Recall that S_n is the group of all bijections or *permutations* on $[n] := \{1, \ldots, n\}$.

Definition 2.2. A *partition* λ of a positive integer n is defined as a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of positive integers, called *parts*, such that $\sum_{i=1}^{\ell} \lambda_i = n$. It is denoted as $\lambda \vdash n$ or $n = |\lambda|$.

Definition 2.3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$. The *Young diagram* of a partition λ is an array of ℓ left-justified rows of boxes with λ_i boxes in the i^{th} row. The following is a Young diagram of shape $(5, 3, 1) \vdash 9$.

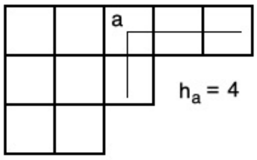


Definition 2.4. A *standard Young tableau* or an SYT of a shape λ is a filling of a Young diagram of λ with integers $1, 2, \ldots, n$ in such a way that the entries are strictly increasing along each row and column.

Example 2.5. Let $\lambda = (5, 3, 1)$ be a partition of 9. The following is an example of a standard Young tableau of shape λ.

1	2	5	6	9
3	7	8		
4				

Definition 2.6. The *hook* of a box a in a Young diagram is defined as the set of boxes directly to its right and directly below it, including itself. The *hook length* of a, denoted by h_a, is the number of boxes in the hook of a.



Example 2.7. The following is the Young diagram of shape $\lambda = (5, 3, 1)$ with the respective hook lengths stated in each box.

7	5	4	2	1
4	2	1		
1				

The set of all standard Young tableaux of a given shape λ is denoted by $\mathrm{tab}(\lambda)$ and the cardinality of this set is denoted by f_λ. For example, $f_{(3,2)} = 5$ and all the standard Young tableaux of shape $(3, 2)$ are shown in Figure 2.2 below.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$$

Figure 2.2: Standard Young tableaux of shape $(3, 2)$

This number can be counted using the following result by Frame, Robinson and Thrall which is known as the *hook-length formula.*

Theorem 2.8. *[60] Let $\lambda \vdash n$ be an integer partition. The number of standard Young tableaux of a shape λ is given by*

$$f_\lambda = \frac{n!}{\Pi_{a\in\lambda} h_a}, \tag{2.2.1}$$

where the product is over all the boxes in the Young diagram of λ and h_a is the hook length of box a.

We now calculate the hook length formulae of two-row and three-row Young diagrams as we come across these shapes frequently in our analysis in this thesis. For $\lambda = (a, b)$, the hook lengths of each of the boxes are depicted as:

$a+1$	$\dots$	$a-b+2$	$a-b$	$\dots$	1
b	$\dots$	1			

Thus, the number of standard Young tableaux of shape λ is given by

$$f_{(a,b)} = \frac{(a+b)!(a-b+1)}{(a+1)!b!} = \frac{a-b+1}{a+1}\binom{a+b}{a}. \tag{2.2.2}$$

Similarly for $\mu = (a, b, c)$, we have

$a{+}2$	$\ldots$	$a{-}c{+}3$	$a{-}c{+}1$	$\ldots$	$a{-}b{+}2$	$a{-}b$	$\ldots$	1
$b{+}1$	$\ldots$	$b{-}c{+}2$	$b{-}c$	$\ldots$	1			
c	$\ldots$	1						

$$\begin{aligned} f_{(a,b,c)} &= \frac{(a+b+c)!(a-c+2)(a-b+1)(b-c+1)}{(a+2)!(b+1)!c!} \\ &= \frac{(a-c+2)(a-b+1)(b-c+1)}{(a+2)(a+1)(b+1)}\binom{a+b+c}{a,b,c}. \end{aligned} \tag{2.2.3}$$

Remark 2.9. It is straightforward to verify from (2.2.2), that $f_{(a,b)}$ satisfies an interesting recurrence relation given by:

$$f_{(a,b)} = f_{(a-1,b)} + f_{(a,b-1)}.$$

Definition 2.10. Given two partitions λ, μ such that $\mu \subseteq \lambda$ (containment order, i.e., $\mu_i \leq \lambda_i$ for all i), the *skew shape* λ/μ is a Young diagram that is obtained by subtracting the Young diagram of shape μ from that of λ.

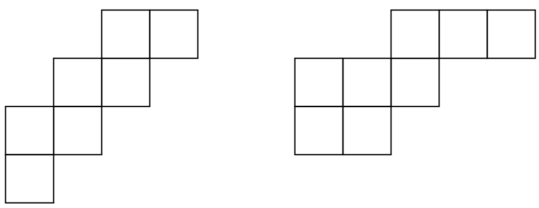


Figure 2.3: Skew shapes $(4, 3, 2, 1)/(2, 1)$ and $(5, 3, 2)/(2)$ respectively

Definition 2.11. A *standard Young tableau of a skew shape* λ/μ is a filling of the Young diagram of the skew shape by positive integers that are strictly increasing in rows and columns.

Example 2.12. Following are a few of the many standard Young tableaux of the same shape $(4, 3, 2, 1)/(2, 1)$.

$$\begin{ytableau} \none & \none & 1 & 4 \\ \none & 2 & 3 \\ 5 & 6 \\ 7 \end{ytableau} \qquad \begin{ytableau} \none & \none & 1 & 2 \\ \none & 5 & 7 \\ 3 & 6 \\ 4 \end{ytableau} \qquad \begin{ytableau} \none & \none & 2 & 3 \\ \none & 1 & 7 \\ 4 & 6 \\ 5 \end{ytableau}$$

Let $f_{\lambda/\mu}$ denote the number of SYTs of a skew shape λ/μ. This number is counted using the Frobenius' determinant formula as follows.

Theorem 2.13. *[95, Corollary 7.16.3] Let $|\lambda/\mu| = n$ be the number of boxes in the skew shape λ/μ that has ℓ parts. Then,*

$$f_{\lambda/\mu} = n! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1}^{\ell} \tag{2.2.4}$$

Note that we take $0! = 1$ and $k! = 0$ for any $k < 0$ as a convention.

Example 2.14. Let $\lambda/\mu = (6,4)/(3)$. Then, $n = |\lambda/\mu| = 7$ and $\ell = 2$. We have,

$$f_{\lambda/\mu} = 7! \begin{vmatrix} \frac{1}{(6-3-1+1)!} & \frac{1}{(6-0-1+2)!} \\ \frac{1}{(4-3-2+1)!} & \frac{1}{(4-0-2+2)!} \end{vmatrix} = 34.$$

Example 2.15. Let $\lambda/\mu = (5,3,2)/(2)$. Then, $n = |\lambda/\mu| = 8$ and $\ell = 3$. We have,

$$f_{\lambda/\mu} = 8! \begin{vmatrix} \frac{1}{(5-2-1+1)!} & \frac{1}{(5-0-1+2)!} & \frac{1}{(5-0-1+3)!} \\ \frac{1}{(3-2-2+1)!} & \frac{1}{(3-0-2+2)!} & \frac{1}{(3-0-2+3)!} \\ \frac{1}{(2-2-3+1)!} & \frac{1}{(2-0-3+2)!} & \frac{1}{(2-0-3+3)!} \end{vmatrix} = 260.$$

Chapter 3

Correlations in the continuous multispecies TASEP on a ring

3.1 Introduction

Multispecies exclusion processes and their different properties have been a popular topic of investigation in recent times. One property which is of great interest is the correlation of two or more particles in the stationary distribution of the process. In this chapter, we prove a result of correlations of two adjacent points in a multispecies TASEP on a continuous ring. We will give the exact definitions in Section 3.2.

One of the first instances where the continuous multispecies TASEP on a ring was mentioned is by Aas and Linusson [4]. They studied a distribution which should be a certain infinite limit of the stationary distribution of the multispecies TASEP on a ring. They also conjectured [4, Conjecture 4.2] a formula for correlations $c_{i,j}$ which is given by the probability that the two particles, labelled i and j are next to each other with i on the left of j in the limit distribution. We prove it first for the case $i > j$ (Theorem 3.1) in Section 3.3 and for then the case $i < j$ (Theorem 3.2) in Section 3.4. The technique we use is similar to and inspired by the work of Ayyer and Linusson [20] where they study correlations in the multispecies TASEP on a ring with a finite number of sites.

To carry out the analysis, we use the theory of the *multiline process* that Ferrari and Martin described in [57]. The multiline process is defined on structures known as *multiline queues* or MLQs. This process can be projected to the multispecies TASEP using a procedure known as *lumping of chains* (see [74, Lemma 2.5]). This projection lets us study the stationary distribution of the multiline process to infer results for the stationary distribution of the multispecies TASEP and is defined using an algorithm known as *bully path projection* which projects a multiline queue to a word. See Section 3.2

for the precise definitions.

We study the two-point correlations in a continuous TASEP with type $\langle 1^{\mathrm{n}} \rangle = \underbrace{(1, \dots, 1)}_{n}$. Note that each particle has a distinct label in this case. In this regard, let $c_{i,j}(n)$ denote the correlation of two particles of types i and j lying adjacent on a ring for the multispecies TASEP with type $\langle 1^{\mathrm{n}} \rangle$, with i followed by j. Aas and Linusson gave an explicit conjecture ([4, Conjecture 4.2]) for calculating $c_{i,j}(n)$. We prove their conjecture in this chapter separately for the two cases.

Theorem 3.1. *For $n \geq 2$ and $i > j$, we have the following two-point correlations:*

$$c_{i,j}(n) = \begin{cases} \frac{n}{\binom{n+j}{2}} - \frac{n}{\binom{n+i}{2}}, & \text{if } j < i < n, \\ \frac{n(j+1)}{\binom{n+j}{2}} - \frac{n(j-1)}{\binom{n+j-1}{2}} - \frac{n}{\binom{2n}{2}}, & \text{if } j < i = n. \end{cases}$$

Theorem 3.2. *For $n \geq 2$ and $i < j$, we have the following two-point correlations:*

$$c_{i,j}(n) = \begin{cases} \frac{n}{\binom{n+j}{2}}, & \text{if } i + 1 < j \leq n, \\ \frac{n}{\binom{n+j}{2}} + \frac{ni}{\binom{n+i}{2}}, & \text{if } i + 1 = j \leq n. \end{cases}$$

3.2 Preliminaries

A multispecies TASEP is a stochastic process on a graph. We first define the multispecies TASEP on a ring before proceeding to study the continuous multispecies TASEP on a ring.

3.2.1 Multispecies TASEP

A multispecies TASEP is a continuous-time Markov process which can be defined on a ring with L sites. For a tuple $m = (m_1, \dots, m_n)$, a multispecies TASEP of type m has $m_1 + \cdots + m_n$ sites occupied with particles. Each particle is assigned a label from the set $[n]$ and there are exactly m_k particles with the label k. The unoccupied sites are treated as particles with the highest label $n + 1$. The states of the multispecies TASEP are words of length L with the letter k occurring m_k times for all $k \in [n]$ and $n+1$ occurring $L - \sum_i m_i$ times. The dynamics of the process are as follows: Each particle carries an exponential clock that rings with rate 1. The particle tries to jump to the site on its left whenever the clock rings. Let this particle be labelled i. The jump is successful only if the site on the left has a label greater than i. In that case, the two particles exchange positions. In other words, let the current state be π with $\pi_k = i$ and $\pi_{k+1} = j$ for some k. The particles labelled i and j interchange positions with rate 1 if $i > j$.

Now we define a Markov process known as the *multiline process* which can be projected to the multispecies TASEP through lumping. A multiline process is a Markov process defined on a graph which is a collection of disjoint cycle graphs of the same length. We refer to each path graph as a row and the rows are numbered from top to bottom. Each row has the same number of sites and each site may or may not be occupied by a particle. For an n-tuple $m = (m_1, \dots, m_n)$ with $m_k \geq 0$ and $L \geq m_1 + \cdots + m_n$, a *multiline queue* of type m is a collection of n rows, each having L sites, stacked on top of each other. In the i^{th} row from the top, $S_i = m_1 + \cdots + m_i$ of the sites are occupied. See Figure 3.1 for an example of a multiline queue. The dynamics of the multiline process

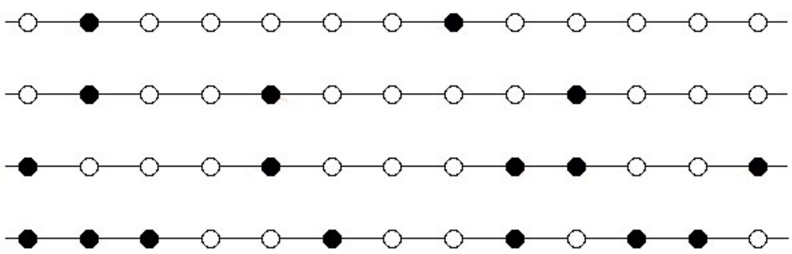


Figure 3.1: A multiline queue of type $(2, 1, 2, 2)$ on 13 sites.

are described in detail in [57] via transitions on the multiline queues of a fixed type. The stationary distribution of the process is stated in the following theorem.

Theorem 3.3. *[57, Theorem 3.1] The stationary distribution of the multiline process of type m is uniform.*

A multiline queue of type m can be projected to a word by an algorithm known as the *bully path procedure* which we define recursively as follows:

(1) Let M be a multiline queue of type m. We construct *bully paths* that contain exactly one particle from each row. Start with the first row in M. The bully path starting at any particle in the first row moves downwards and then rightwards along the multiline queue until it runs into a particle in the second row. It again moves downwards and rightwards in the third row till it hits another particle, and so on all the way to the last row. All the particles encountered by this bully path are labelled 1. We similarly construct the bully paths starting from other particles in the first row. It turns out that the order in which these particles are constructed starting from the particles in

the first row does not matter. At the end of this construction, we have a total of m_1 bully paths. That is, m_1 particles of the last row are labelled 1. See Figure 3.2 for the construction of bully paths to the multiline queue in Figure 3.1.

(2) Next, we construct bully paths starting with the unlabelled particles in the second row by repeating the same process from Step (1). We label the ends of all such paths as 2 and they are m_2 in number. We repeat these steps for all the subsequent rows. Finally, label all the particles that are left unlabelled in the last row as n and all the unoccupied sites as $n+1$.

(3) Hence for each ℓ, m_ℓ particles in the last row are labelled as ℓ. Let ω denote the word formed by the labels in the last row. Then, ω is a configuration of the multispecies TASEP of type m. Let $\mathbf{B}$ denote this projection map. Then, ω is known as the *projected word* of M and we write it as $\omega = \mathbf{B}(M)$. The projected word for the multiline queue in Figure 3.1 is 3345515525145; see Figure 3.2.

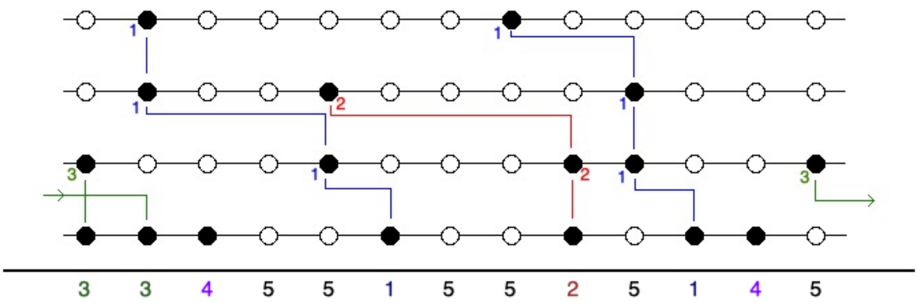


Figure 3.2: Bully path projection on the multiline queue from Figure 3.1. The bully paths starting in the first, second and third rows are shown in colours blue, red and green respectively.

The connection between the stationary distributions of the multiline process and the multispecies TASEP is established by the following theorem given by Ferrari and Martin [57].

Theorem 3.4. *[57, Theorem 4.1] The process on the last row of the multiline process of type m is the same as the multispecies TASEP of type m.*

3.2.2 Continuous multispecies TASEP

Fix $m = (m_1, \ldots, m_n)$ and let $S_i = m_1 + \cdots + m_i$, for all $i \in [n]$. The continuous multispecies TASEP is a formal limit of the stationary distribution of the multispecies

TASEP on a ring with L sites. First, we consider a multispecies TASEP on a ring with L sites. Let Π_n^L denote the stationary distribution of this TASEP. Then, L is taken to infinity keeping the vector m constant, i.e., the number of unoccupied sites tends to infinity. The ring is then scaled to the continuous interval $[0, 1)$. The limit of the stationary distribution Π_n^L gives a distribution Π_n of labelled particles on a continuous ring. Note that Π_n is not yet shown to be the stationary distribution of any Markov process yet.

Similar to the multiline queues in [57], we can look at the continuous multiline queues of a given type. For $m = (m_1, \ldots, m_n)$ and $S_i = m_1 + \cdots + m_i$, for all $i \in [n]$, a *continuous multiline queue* of type m is defined on a collection of n copies of $[0, 1)$ stacked on top of each other with S_i particles in i^{th} ring from the top.

The location of each particle is considered to be a real number in the continuous interval $[0, 1)$. In the distribution that we will consider, the horizontal position of each particle will almost surely be distinct.

Example 3.5. See Figure 3.3 for an example of a continuous multiline queue of type $(1, 3, 1, 2)$. The rows have $1, 4, 5$ and 7 particles respectively. Ignore the labels of the particles for now. Note that there is no particle directly above or below any other particle.

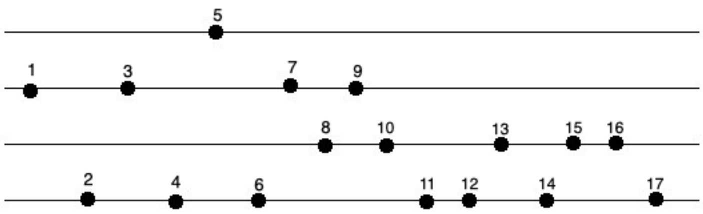


Figure 3.3: A continuous multiline queue of type $(1, 3, 1, 2)$

Consider the labels of the particles in Figure 3.3. The labels are assigned in the order of the horizontal positions of the particles as seen from left to right. We now refer to an integer representation of a continuous multiline queue which is also used by Aas and Linusson [4].

Definition 3.6. Let $m = (m_1, \ldots, m_n)$ be an n-tuple, let $S_i = m_1 + \cdots + m_i$, and let $N = \sum_{i=1}^{n} S_i$. A *placement* of a continuous multiline queue of type m is a triangular array $(p_{i,k})$ with distinct integers from the set $[N]$ such that the integer $p_{i,k}$ stands for the relative horizontal position of k^{th} particle in the i^{th} row of the multiline queue as seen from left to right.

Example 3.7. The placement corresponding to the continuous multiline queue in Figure 3.3 is

$$\begin{array}{lllllll} 5 \\ 1 & 3 & 7 & 9 \\ 8 & 10 & 13 & 15 & 16 \\ 2 & 4 & 6 & 11 & 12 & 14 & 17. \end{array}$$

For our purpose, it is enough to know the relative positions of the particles on the rows and hence a continuous multiline queue will be represented by its placement and we will use the two terms interchangeably. The number of different placements of type m is given by $\binom{N}{S_1,\dots,S_n}$.

The *bully path projection* is a map from the set of continuous multiline queues of type m to the words of type m. Let M be a continuous multiline queue. We define the algorithm recursively which takes M and maps it to a word ω as follows:

(1) Consider the placement of M. For each integer k_1 in the first row, look for the smallest available entry larger than k_1 in the second row and mark it as k_2. If k_1 is larger than all the available integers in the second row, we mark the smallest available integer in the second row as k_2. This is known as *wrapping* from the first row to the second row. We say that k_1 "bullies" k_2 and write it as $k_1 \to k_2$; or $k_1 \xrightarrow{W} k_2$ in the case of wrapping. See Figure 3.4 for the map applied to Example 3.7.

(2) Look for the smallest available integer in the third row larger than k_2 and label it k_3 and so on. The sequence $k_1, k_2, \dots, k_n$ thus obtained is called a *bully path* starting at k_1 and these integers are now unavailable for further bullying. Label the endpoints of all such paths with 1. There are m_1 such paths and we call them *type* 1 bully paths. In Figure 3.4, $5 \to 7 \to 8 \to 11$ is a bully path of type 1. The order in which we construct the bully paths starting in the first row does not matter.

(3) Next, we construct bully paths starting with the available integers in the second row by following steps (1) and (2). We label the ends of all such paths with 2 and they are m_2 in number. We repeat these steps for all the other rows sequentially. The bully paths of type n are just the integers in the last row that are remaining after the construction of all type $(n-1)$ bully paths.

(4) Therefore, there are m_ℓ bully paths of type ℓ for each ℓ. Let ω denote the word formed by the labels in the last row. Then, ω gives the relative ordering of labelled

particles in a configuration of the continuous multispecies TASEP on a ring. Let $\mathbf{B}$ denote this projection map. Then, $\omega = \mathbf{B}(M)$ is known as the *projected word* of M. The projected word for the continuous multiline queue in Example 3.7 is 3441222; see Figure 3.4.

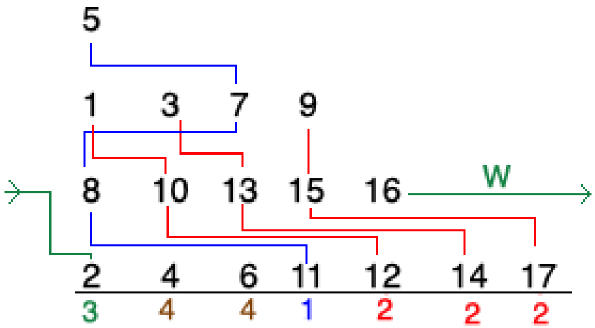


Figure 3.4: Bully path procedure on a continuous multiline queue of type (1,3,1,2). The type $1, 2$ and 3 bully paths are shown in colours blue, red and green respectively.

The distribution of the words of type m is the same as the distribution of the last row of continuous multiline queues of type m which can be obtained by taking the limit of the distribution of the last row for discrete multiline queues. Also, the distribution of the last row for discrete multiline queues is Π_n^L by Theorem 3.4. Therefore, Π_n gives the distribution of the labels on the last row for a uniformly chosen continuous multiline queue. Thus, to study the correlations of two adjacent particles with labels i and j in the continuous multispecies TASEP, it is enough to count the number of placements that project to the words with i and j as adjacent particles. Next, we define an operator on the space of all continuous multiline queues of a fixed type.

Definition 3.8. Let M be a continuous multiline of type m. Let S be an operator such that if $M' = S(M)$, then M' is obtained from M by adding $1 \mod N$ to each entry of M and then rearranging any row in increasing order if needed. S is known as the *shift operator* and we will refer to M' as the *shifted* multiline queue.

Example 3.9. Let M be the multiline queue from Example 3.7, then $S(M) = M'$ is given by

$$M' = \begin{array}{lllllll} 6 & & & & & & \\ 2 & 4 & 8 & 10 & & & \\ 9 & 11 & 14 & 16 & 17 & & \\ 1 & 3 & 5 & 7 & 12 & 13 & 15. \end{array}$$

The following lemma relates the projected words of a continuous multiline queue and its shifted continuous multiline queue.

Lemma 3.10. *Let M be a continuous multiline queue of type m with $N = \sum_{i=1}^{n} S_i$ particles. Shifting M rotates the projected word by one position to the right when the largest entry N is in the last row of the placement and preserves the word otherwise. In other words, if $M' = S(M)$, $\omega = \mathbf{B}(M)$ and $\omega' = \mathbf{B}(M')$, then ω and ω' are related in the following way:*

1. *If N is not in the last row, then $\omega' = \omega$.*

2. *If N is in the last row, then ω' is obtained by rotating ω one position to the right.*

Proof. Let N be in the r^{th} row of the placement of M. First, let $r < n$. M' is obtained from M by replacing N with 1 and adding 1 to every other integer. By the increasing property of the rows, the r^{th} row is now rotated by one position to the right. If N lies on a bully path that starts in some row above r, then all the bully paths remain the same. This is true because N wraps around and bullies the first entry in $(r+1)^{st}$ row available to it. Whereas in M', 1 being the smallest integer bullies the first entry available to it which is exactly the translation of the entry bullied by N in M. The remaining bully paths are the same since the inequalities among all other elements do not change.

On the other hand, let there be a bully path of type r in M that starts at N. Let $s = r+1$ and the available integers in the s^{th} row in M after the construction of all the bully paths of a type less than r be $s_1 < s_2 < \cdots < s_{n_t}$. If $N \xrightarrow{W} s_x$ (say) in M, then observe that there exist elements in r^{th} row namely r_i $(1 \le i \le x-1)$ such that r_1 bullies s_1, r_2 bullies $s_2, \ldots, r_{x-1}$ bullies s_{x-1} in M. In M', the bully paths that begin in a row above the r^{th} are the same as those in M. For a type r bully path, 1 bullies (s_1+1), (r_1+1) bullies $(s_2+1), \ldots, (r_{x-1}+1)$ bullies (s_x+1). The construction of these type r bully paths in M' from here onwards is the same as the construction of those in M. Thus, the projected word remains the same.

Finally, when $r = n$, i.e., when N is in the last row, the last row rotates by one position to the right when adding $1 \mod N$ to each entry in M. The bully paths remain the same and therefore, the projected word which is obtained from the labels of the particles in the last row is rotated by one position to the right. □

Let $\langle 1^{\mathrm{n}} \rangle = (1, \ldots, 1)$ be an n-tuple. Define $c_{i,j}(n) = \mathbb{P}\{\eta_a = i, \eta_{a+1} = j; a \in [n]\}$. To make the analysis of the continuous multispecies TASEP of type $\langle 1^{\mathrm{n}} \rangle$ easy, we use a classical property known as the *projection principle* which states that particles of two consecutive types cannot be distinguished by particles of other types. Thus, identifying two consecutive labels k and $k+1$ defines a natural projection from the n-TASEP onto the $(n-1)$-TASEP. This is a key observation in [11, Section 1]. Therefore, to obtain $c_{i,j}(n)$ for $i > j$, it is enough to find the probability that 4 is followed by a 2 in the projection of the word of the five-species continuous multiline queue with type $m_{i,j} = (j-1, 1, i-j-1, 1, n-i)$.

We can further simplify the task at hand by projecting many such continuous multiline queues to a three-species system. Given a continuous multiline queue with type $\langle 1^{\mathrm{n}} \rangle$, consider its projection to a continuous multiline queue of type $m_{s,t} = (s, t, n-s-t)$ where $j > s$ and $i > s+t$. Thus, a particle of class j becomes a 2 and that of class i becomes a 3 whenever $t, n-s-t > 0$. So, to compute the correlation of particles labelled i and j in a multispecies TASEP of type $\langle 1^{\mathrm{n}} \rangle$, we need to look at the correlation of particles labelled 3 and 2 in the projected words of continuous multiline queues of type $m_{s,t}$. Similarly, to formulate the correlation of particles labelled i and j for $i < j$, we need to look at the correlation of 2 and 3 in the projected words of continuous multiline queues with type $m_{s,t}$.

Let M be a continuous multiline queue of type $\langle 1^{\mathrm{n}} \rangle$ and $\eta = \mathbf{B}(M)$ be the word that is projected from M using bully path projection. For $1 \leq i, j \leq n$, let $E^a_{i,j}(n) = \mathbb{P}\{\eta_a = i, \eta_{a+1} = j\}$ for $a \in [n]$. Thus,

$$c_{i,j} = \sum_{a=1}^{n} E^a_{i,j}. \tag{3.2.1}$$

Consider a continuous multiline queue M' of type $m_{s,t}$. Let the projected word of M' be ω. If

$$\begin{aligned} \Theta^<_a(s,t) &= \mathbb{P}\{\omega_a = 2, \omega_{a+1} = 3\}, \\ \text{and } \Theta^>_a(s,t) &= \mathbb{P}\{\omega_a = 3, \omega_{a+1} = 2\} \end{aligned}$$

for $a \in [n]$, then by projection principle, we have

$$\begin{aligned} \Theta^<_a(s,t) &= \sum_{j=s+t+1}^{n} \sum_{i=s+1}^{s+t} E^a_{i,j}, \\ \text{and } \Theta^>_a(s,t) &= \sum_{i=s+t+1}^{n} \sum_{j=s+1}^{s+t} E^a_{i,j}. \end{aligned}$$

Let $\mathcal{T}^<$ (respectively $\mathcal{T}^>$) be the sum of $\Theta_a^<$ (respectively $\Theta_a^>$) over the index a. We have,

$$\begin{aligned}\mathcal{T}^<(s,t) &= \sum_{a=1}^{n}\sum_{j=s+t+1}^{n}\sum_{i=s+1}^{s+t} E_{i,j}^a(n)\\ &= \sum_{j=s+t+1}^{n}\sum_{i=s+1}^{s+t} c_{i,j}(n).\end{aligned}$$

The last equality follows from (3.2.1). Similarly,

$$\mathcal{T}^>(s,t) = \sum_{i=s+t+1}^{n}\sum_{j=s+1}^{s+t} c_{i,j}(n).$$

For $i < j$, the principle of inclusion-exclusion then gives us

$$c_{i,j}(n) = \mathcal{T}^<(i-1, j-i) - \mathcal{T}^<(i, j-i-1) - \mathcal{T}^<(i-1, j-i+1) + \mathcal{T}^<(i, j-i), \quad (3.2.2)$$

and for $i > j$, we have

$$c_{i,j}(n) = \mathcal{T}^>(j-1, i-j) - \mathcal{T}^>(j, i-j-1) - \mathcal{T}^>(j-1, i-j+1) + \mathcal{T}^>(j, i-j). \quad (3.2.3)$$

For $a \in [n]$, if we let

$$\begin{aligned} T_a^<(s,t) &= \mathbb{P}\{\omega_a = 2, \omega_{a+1} = 3, M'_{3,n} = N\},\\ \text{and } T_a^>(s,t) &= \mathbb{P}\{\omega_a = 3, \omega_{a+1} = 2, M'_{3,n} = N\},\end{aligned}$$

then the following lemma holds.

Lemma 3.11. $\mathcal{T}^<(s,t) = (n + 2s + t)\, T_a^<(s,t)$ *for any* $a \in [n]$.

Proof. Let M' be a continuous multiline queue of type $m_{s,t}$. Shifting it $N = n + 2s + t$ times generates all the rotations of the projected word. By Lemma 3.10, in exactly n out of N shifted continuous multiline queues, the projected word rotates one unit to the right and in the remaining shifts, the projected word remains the same. For a fixed $a \in [n]$, any continuous multiline queue which contributes to $\mathcal{T}^<(s,t)$ can be obtained as a rotation of a continuous multiline queue for which

(i) the projected word has 3 and 2 (or 2 and 3) in positions a and $a+1 \mod n$ respectively, and

(ii) N is in the last row.

Note that if a word has a 3 followed by a 2 in p separate positions, it occurs as a rotation of p different words with $\omega_a = 3, \omega_{a+1} = 2$. Hence, the result. $\square$

We also have $\mathcal{T}^{>}(s,t) = (n+2s+t)T_a^{>}(s,t)$ for any $a \in [n]$. Further, Lemma 3.11 holds for $a = 1$ in particular. Henceforth, we will write $T_1^{<}$ (respectively $T_1^{>}$) as $T^{<}$ (respectively $T^{>}$) for simplicity. Then, (3.2.2) and (3.2.3) become

$$\begin{aligned} c_{i,j}(n) =& (n+i+j-2)\, T^{<}(i-1, j-i) - (n+i+j-1)\, T^{<}(i, j-i-1) \\ & - (n+i+j-1)\, T^{<}(i-1, j-i+1) + (n+i+j)\, T^{<}(i, j-i), \end{aligned} \tag{3.2.4}$$

and

$$\begin{aligned} c_{i,j}(n) =& (n+i+j-2)\, T^{>}(j-1, i-j) - (n+i+j-1)\, T^{>}(j, i-j-1) \\ & - (n+i+j-1)\, T^{>}(j-1, i-j+1) + (n+i+j)\, T^{>}(j, i-j) \end{aligned} \tag{3.2.5}$$

respectively.

Lemma 3.12. *Let $m = (m_1, \ldots, m_n)$ be a tuple and let $S_i = m_1 + \cdots + m_i$, for all $i \in [n]$. The set of continuous multiline queues of type m that have no wrapping under the bully path projection is in bijection with standard Young tableaux of shape λ where $\lambda_i = S_{n-i+1} = m_1 + \cdots + m_{n-i+1}$.*

Proof. Let M be a continuous multiline queue of type m such that $N = S_1 + \cdots + S_n$ is the largest integer in M. For $1 \le i \le i, 1 \le j \le S_i$, let $M_{i,j}$ be distinct integers from 1 to N. Then, we have

$$M = \begin{array}{llllll} M_{1,1} & \ldots & M_{1,S_1} & & & \\ M_{2,1} & \ldots & \ldots & M_{2,S_2} & & \\ \vdots & & & & & \\ M_{n,1} & \ldots & \ldots & & \ldots & M_{n,S_n}. \end{array}$$

No wrapping from the first row to the second row implies $M_{1,S_1-k} < M_{2,S_2-k}$ for all $0 \le k \le S_1 - 1$. Similarly, $M_{2,S_2-k} < M_{3,S_3-k}$ for all $0 \le k \le S_2 - 1$, and so on. This along with the increasing property of the rows gives us

$$\begin{array}{ccccccccccccccc}
 & & & & M_{1,1} & < & \cdots & < & M_{1,S_1-i} & < & \cdots & < & M_{1,S_1} \\
 & & & & \wedge & & & & \wedge & & \cdots & & \wedge \\
 & & M_{2,1} & < & \cdots & < & \cdots & < & M_{2,M=S_2-i} & < & \cdots & < & M_{2,S_2} \\
 & & \wedge & & & & & & \wedge & & & & \wedge \\
 & & & & & & & & & & & & \vdots \\
M_{n,1} & < & \cdots & < & \cdots & < & \cdots & < & M_{n,S_n-i} & < & \cdots & < & M_{n,S_n}.
\end{array}$$

Hence, M is in bijection with a standard Young tableau $\mathcal{Y}$ of shape $(S_n, \ldots, S_1)$ and the bijection is given by $\mathcal{Y}_{i,j} = N - M_{n-i+1,S_{(n-i+1)}-j+1} + 1$. Thus, the number of continuous multiline queues of type m with no wrapping is given by $f_{(S_n,\ldots,S_1)}$. □

3.2.3 Two species continuous TASEP

Let $m = (s, t)$. In this section, we study the continuous TASEP with two kinds of particles and a hole. Because of the simplicity of the structure, it is easy to completely calculate the stationary distribution of the continuous TASEP on two species. Let the correlation $c_{i,j}(n)$ be defined as the probability that the particle labelled i is immediately followed on the ring by a particle labelled j in the limit distribution. Once again, we use the continuous multiline queues of type m to find these correlations.

Let $\Omega_{s,t}$ be the set of continuous multiline queues of type (s, t) and let $n = s + t$. That is, $M \in \Omega_{s,n-s}$ is a continuous multiline queue that has s integers in the first row and n integers in the second row. For $c \in \{1, 2\}$, let $\delta_c(s, n)$ be the number of continuous multiline queues of $\Omega_{s,n-s}$ with the largest integer $n + s$ in the second row such that the projected word has c in the first position. By the well-known property of rotational symmetry of the multispecies TASEP (see [20, Proposition 2.1 (iii)]), we have

$$\delta_1(s, n) = \frac{s}{n}\binom{n+s-1}{s}, \tag{3.2.6}$$

$$\delta_2(s, n) = \frac{n-s}{n}\binom{n+s-1}{s}. \tag{3.2.7}$$

Remark 3.13. Note that the number of configurations of $\Omega_{s,n-s}$ with $n+s$ in the second row such that the projected word has c in the a^{th} position is the same for any $a \in [n]$ by rotational symmetry.

Similarly, let $\delta_{c,d}(s, n)$ count the number of continuous multiline queues of type $(s, n-s)$ that have the largest integer $n+s$ in the second row such that the projected word has

c in the first place and d in the second place. Using Remark 3.13, we have the following system of independent equations for fixed s and n.

$$\delta_{1,1} + \delta_{1,2} = \delta_1, \tag{3.2.8}$$

$$\delta_{1,1} + \delta_{2,1} = \delta_1, \tag{3.2.9}$$

$$\delta_{1,2} + \delta_{2,2} = \delta_2. \tag{3.2.10}$$

Therefore, finding $\delta_{c,d}$ for any $(c, d) \in \{1, 2\} \times \{1, 2\}$ solves the system of equation. In particular, let $c = d = 2$. Consider an arbitrary continuous multiline queue M' such that $\pi = \mathbf{B}(M')$ with the following configuration:

$$\begin{array}{c|cccccc}
 & a_1 & a_2 & \dots & a_s & & \\
 & b_1 & b_2 & \dots & \dots & b_{n-1} & n+s \; . \\
\hline
\pi = & 2 & 2 & \dots & \dots & \dots & \dots
\end{array}$$

Since $\pi_1 = \pi_2 = 2$, b_1 and b_2 are not bullied by any a_i in M'. Therefore, $a_1 > b_2$. We also have $b_2 > b_1$ by the increasing property of the rows. This forces $b_1 = 1$ and $b_2 = 2$. Moreover, there is no wrapping from the first row to the second row. This implies that the integers in M' other than b_1 and b_2 satisfy the following inequalities:

$$\begin{array}{ccccccccccc}
 & & & & a_1 & < & \cdots & < & a_{s-1} & < & a_s \\
 & & & & \wedge & & \cdots & & \wedge & & \wedge \\
b_3 & < & \cdots & < & b_{n-s+1} & < & \cdots & < & b_{n-1} & < & n+s.
\end{array}$$

Such configurations are in bijection with standard Young tableaux of shape $(n-2, s)$ by Lemma 3.12. Therefore using (2.2.2), we get

$$\delta_{2,2}(s, n) = f_{(n-2,s)} = \frac{n-s-1}{n+s-1}\binom{n+s-1}{s}. \tag{3.2.11}$$

Using (3.2.6)-(3.2.11), we can solve for all the remaining $\delta_{c,d}$ as follows:

$$\delta_{1,2}(s, n) = \delta_{2,1}(s, n) = \frac{n-s+1}{n+s-1}\binom{n+s-1}{s-1}, \tag{3.2.12}$$

$$\delta_{1,1}(s, n) = 2\binom{n+s-2}{s-2}. \tag{3.2.13}$$

3.3 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 using the tools developed in Section 3.2. Let $m_{s,t} = (s,t,n-s-t)$ and let $\mathcal{S}_{s,t}$ be the set of all continuous multiline queues M of type $m_{s,t}$ that satisfy $\omega_1 = 3$ and Let $m_{s,t} = (s,t,n-s-t)$ and let $\mathcal{S}_{s,t}$ be the set of all continuous multiline queues M of type $m_{s,t}$ that satisfy $\omega_1 = 3$ and $\omega_2 = 2$ where $\omega = \mathbf{B}(M)$. Recall that $T^{>}(s,t) = \mathbb{P}\{\omega_1 = 3, \omega_2 = 2, M_{3,n} = N\}$. We first compute $T^{>}(s,t)$ and then substitute it in (3.2.3) to solve for $c_{i,j}$ for the case $i > j$. Let $\tau_{s,t}$ denote the cardinality of the set $\mathcal{S}_{s,t}$. That is, $\tau_{s,t}$ counts the continuous multiline queues which have the following structure where a_i, b_i and c_i be distinct integers from the set $[N]$.

$$\begin{array}{c|cccccccc}
 & a_1 & a_2 & \dots & \dots & a_s & & & \\
 & b_1 & b_2 & \dots & b_k & \dots & b_{s+t} & & \\
 & c_1 & c_2 & \dots & \dots & \dots & \dots & c_n = N. \\
\hline
\omega = & 3 & 2 & \dots & \dots & \dots & \dots & \dots
\end{array}$$

Lemma 3.14. *Let M be a continuous multiline queue of type $m_{s,t}$. $M \in \mathcal{S}_{s,t}$ if and only if the following conditions hold:*

(1) If $a_1 \to b_k$, then $b_k > c_2$,

(2) $c_1 = 1$ and $b_1 = 2$,

(3) there is no wrapping from any of the two rows.

Proof. Let $M \in \mathcal{S}_{s,t}$ and suppose that $a_1 \to b_k$. If $b_k < c_2$, then b_k bullies either c_1 or c_2 and we get $\omega_1 = 1$ or $\omega_2 = 1$. Hence, (1) holds. If there is any wrapping from the second row to the third row, then we have $\omega_1 = 1$ or 2, which is a contradiction. Further $\omega_2 = 2$ implies that $b_1 \to c_2$ and b_1 is not bullied by any a_i. Therefore, $c_1 < b_1 < c_2$ and $b_1 < a_1$, and hence $c_1 = 1$ and $b_1 = 2$.

If there is any wrapping from the first row to the second row, then we have $a_i \to b_1$ for some i, which is a contradiction, thus proving the remaining half of 3. It is straightforward to verify that the three conditions imply $\omega_1 = 3$, $\omega_2 = 2$ and $c_n = N$. □

Proof of Theorem 3.1. Let $M \in \mathcal{S}_{s,t}$. We have $c_1 = 1$, $b_1 = 2$ and $c_n = N$ and we have to find the number of ways the remaining entries of M can be assigned values from the set $\{3, \dots, N-1\}$ complying with the three conditions of Lemma 3.14.

Any b_ℓ which is bullied by some a_m should be larger than c_2. Thus, there are at least s integers in the second row that are larger than c_2. Therefore, $c_2 \in \{3, 4, \dots, s+t+2\}$ as there can be at most s integers in the first row and at most t integers in the second row that are less than c_2. Let $c_2 = i$ and let $a_1 \to b_k$. Then, $c_2 < b_k$ and $b_{k-1} < a_1 < b_k$. So, all the numbers from 3 to $i-1$ are assigned, in order, to $b_2 < \cdots < b_{k-1} < a_1 < \cdots < a_{i-k-1}$. Therefore, $0 \le i-k-1 \le s$, i.e., $i-s-1 \le k \le i-1$. Also, since $k-1$ is the number of entries in the second row that are smaller than c_2, it is bounded by 1 and t. Hence, $2 \le k \le t+1$.

The ways in which the remaining entries can be assigned values are in bijection with standard Young tableaux of shape $\lambda_{i,k} = (n-2, s+t-k+1, s-i+k+1)$ by Lemma 3.12 because there is no wrapping from any of the rows. This gives us

$$\tau_{s,t} = \sum_{i=3}^{s+t+2} \sum_{k=\max\{2,i-s-1\}}^{\min\{i-1,t+1\}} f_{\lambda_{i,k}}, \tag{3.3.1}$$

where f_λ is the number of standard Young tableaux of shape λ. By the hook length formula for a 3-row partition (2.2.3), we have:

$$\begin{aligned} f_{\lambda_{i,k}} &= \frac{(n+2s+t-i)!(n-s+i-k-1)(n-s-t+k-2)(t+i-2k+1)}{n!(s+t-k+2)!(s-i+k+1)!} \\ &= \frac{(n-s+i-k-1)(n-s-t+k-2)(t+i-2k+1)}{(n+2s+t-i+3)(n+2s+t-i+2)(n+2s+t-i+1)} \\ &\quad \times \binom{n+2s+t-i+3}{n, s+t-k+2, s+k-i+1}. \end{aligned}$$

Substituting this formula in (3.3.1) and changing k to $k' = k-2$ and i to $j = i-3$, we get

$$\begin{aligned} \tau_{s,t} = \sum_{j=0}^{s+t-1} \sum_{k'=\max\{j-s,0\}}^{\min\{j,t-1\}} &\frac{(n-s+j-k')(n-s-t+k')(t+j-2k')}{(n+2s+t-j)(n+2s+t-j-1)(n+2s+t-j-2)} \\ &\times \binom{n+2s+t-j}{n, s+t-k', s-j+k'}. \end{aligned} \tag{3.3.2}$$

Note that when $j \ge s$ and $k' \le j-s$, the multinomial coefficient in (3.3.2) becomes 0. Therefore, the index k' can equivalently be summed over the range 0 to $\min\{j, t-1\}$. Substituting v for $s+t-j-1$ and u for $s-j+k'$ in (3.3.2), we get

$$\begin{aligned} \tau_{s,t} &= \sum_{v=0}^{s+t-1} \sum_{u=v-t+1}^{\min\{s,v\}} \frac{(n-u)(n+u-s-v-1)(s+v+1-2u)}{(n+s+v+1)(n+s+v)(n+s+v-1)} \binom{n+s+v+1}{n, s+v+1-u, u} \\ &= \sum_{v=0}^{s+t-1} \sum_{u=v-t+1}^{\min\{s,v\}} \frac{(n+s+v-2)!}{n!(s+v+1)!} \zeta_{s,t}, \end{aligned} \tag{3.3.3}$$

where

$$\zeta_{s,t} = (n-u)(n+u-s-v-1)(s+v+1-2u)\binom{s+v+1}{u}.$$

We expand the expression on the right-hand side in order to write $\zeta_{s,t}$ as a sum of two parts as follows:

$$\begin{aligned}\zeta_{s,t} &= ((n^2 - n(s+v+1) + u(s+v+1-2u)) \\ &\quad \times \left((s+v+1-u)\binom{s+v+1}{u} - u\binom{v+s+1}{u}\right) \\ &= (n^2 - n(s+v+1))(s+v+1)\left(\binom{s+v}{u} - \binom{s+v}{u-1}\right) \\ &\quad + (s+v+1)(s+v)(s+v-1)\left(\binom{s+v-2}{u-1} - \binom{s+v-2}{u-2}\right).\end{aligned}$$

Plugging $\zeta_{s,t}$ in (3.3.3), we have a telescoping sum which computes easily to give

$$\begin{aligned}\tau_{s,t} &= \sum_{v=0}^{s+t-1} \frac{(n+s+v-2)!}{n!(s+v+1)!}(n^2 - n(s+v+1))(s+v+1)\left(\binom{s+v}{s} - \binom{s+v}{v-t}\right) \\ &\quad + \sum_{v=0}^{s+t-1} \frac{(n+s+v-2)!}{n!(s+v+1)!}(s+v+1)(s+v)(s+v-1)\left(\binom{s+v-2}{s-1} - \binom{s+v-2}{v-t-1}\right) \\ &= \sum_{v=0}^{s+t-1} \binom{n+s+v}{s+v}\frac{n^2 - n(s+v+1)}{(n+s+v)(n+s+v-1)}\left(\binom{s+v}{s} - \binom{s+v}{v-t}\right) \\ &\quad + \sum_{v=0}^{s+t-1} \binom{n+s+v-2}{s+v-2}\left(\binom{s+v-2}{s-1} - \binom{s+v-2}{v-t-1}\right)\end{aligned}$$

This simplifies to

$$\tau_{s,t} = \frac{\binom{n+2s+t}{n,s+t,s}}{n+2s+t} \times \frac{nt(s+t)}{(n+s)(n+s+t)}\left(-1 + \frac{s}{n} + \frac{(n+s)(n^2+nt-t-s(s+t)-1)}{(n+s-1)(s+t)(n+s+t-1)}\right). \tag{3.3.4}$$

By substituting $\tau_{s,t} = \binom{n+2s+t}{n,s+t,s}T^{>}(s,t)$, we get

$$\begin{aligned}(n+2s+t)T^{>}(s,t) &= \frac{-nt(s+t)}{(n+s)(n+s+t)} + \frac{st(s+t)}{(n+s)(n+s+t)} \\ &\quad + \frac{nt(n^2+nt-t-s(s+t)-1)}{(n+s-1)(n+s+t)(n+s+t-1)}.\end{aligned} \tag{3.3.5}$$

Let $i < n$. Denote the terms on the right-hand side of (3.3.5) by $A(s,t), B(s,t)$ and $C(s,t)$ respectively. We first compute $A(j-1, i-j) - A(j, i-j-1) - A(j-1, i-j+$

$1) + A(j, i-j) = \mathcal{A}(\text{say})$.

$$\begin{aligned}\mathcal{A} &= \frac{n(i-1)}{n+i-1}\left(\frac{j-i}{n+j-1}+\frac{i-j-1}{n+j}\right)-\frac{ni}{n+i}\left(\frac{j-i-1}{n+j-1}+\frac{i-j}{n+j}\right)\\ &= \frac{n}{2\binom{n+j}{2}}.\end{aligned}$$

If we define $\mathcal{B}$ and $\mathcal{C}$ similarly as $\mathcal{A}$ using the inclusion-exclusion formula, we get $\mathcal{B} = \frac{n}{2\binom{n+j}{2}} - \frac{n}{2\binom{n+i}{2}}$ and $\mathcal{C} = \frac{-n}{2\binom{n+i}{2}}$. Since $c_{i,j} = \mathcal{A}+\mathcal{B}+\mathcal{C}$, we have

$$c_{i,j} = \frac{n}{\binom{n+j}{2}} - \frac{n}{\binom{n+i}{2}}.$$

Note that when $s+t=n$, $T^{>}(s,t)=0$. When $j<i=n$, (3.2.5) becomes

$$c_{i,j} = (n+i+j-2)\,T^{>}(j-1,i-j) - (n+i+j-1)\,T^{>}(j,i-j-1).$$

Let $\mathcal{A}' = A(j-1,n-j) - A(j,n-j-1)$ and define $\mathcal{B}'$ and $\mathcal{C}'$ similarly. Simplifying the equations resulting from substituting the expressions for $A(s,t), B(s,t)$ and $C(s,t)$, we get

$$\begin{aligned}c_{n,j} &= \mathcal{A}'+\mathcal{B}'+\mathcal{C}'\\ &= \frac{-1}{2n-1} - \frac{-2n(n-1)}{(n+j-1)(n+j)} + \frac{2n(n-1)}{(n+j-1)(n+j-2)}\\ &= \frac{n(j+1)}{\binom{n+j}{2}} - \frac{n(j-1)}{\binom{n+j-1}{2}} - \frac{n}{\binom{2n}{2}},\end{aligned}$$

thereby proving Theorem 3.1. $\square$

3.4 Proof of Theorem 3.2

Recall that $m_{s,t} = (s,t,n-s-t)$ and $N = n+2s+t$. Recall from (3.2.4) that to find $c_{i,j}(n)$ for $i<j$, we need to know the probability $T^{<}(s,t) = \mathbb{P}\{\omega_1 = 2, \omega_2 = 3, M_{3,n} = N\}$, where ω is the projected word of a continuous multiline queue M of type $m_{s,t}$.

Let $\mathcal{P}_{s,t}$ be the set of all continuous multiline queues M of type $m_{s,t}$ that satisfy $\omega_1 = 2$, $\omega_2 = 3$ and $M_{3,n} = N$ where $\omega = \mathbf{B}(M)$. Let $\theta_{s,t}$ be the cardinality of set $\mathcal{P}_{s,t}$. First, consider the following continuous multiline queue:

$$\begin{array}{ccccccc} a_1 & a_2 & \cdots & a_s & & & \\ b_1 & b_2 & \cdots & \cdots & b_{s+t} & & \\ c_1 & c_2 & \cdots & \cdots & \cdots & c_{n-1} & N. \\ \hline x & y & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Define $n_{x,y}(s,t,n)$ as the number of continuous multiline queues of type $m_{s,t}$ with N in the last row that project to a word with x and y in the first and the second place respectively as above, where $(x,y) \in \{1,2,3\}^2$. Note that $\theta_{s,t} = n_{23}(s,t,n)$. Also, let $n_z(s,t,n)$ be the number of continuous multiline queues of type $m_{s,t}$ with N in the last row that project to a word with z in the first place. Note that by rotational symmetry of multispecies TASEP in [20, Proposition 2.1 (i)], n_z also gives the number of continuous multiline queues with N in the last row that project to a word with z in the second place. Therefore, we have

$$n_{1,3} + n_{2,3} + n_{3,3} = n_3, \tag{3.4.1}$$

for fixed s,t and n. Again by rotational symmetry, we have

$$n_3(s,t,n) = \frac{n-s-t}{N}\binom{n+2s+t}{n,s,s+t}, \tag{3.4.2}$$

since there are $n-s-t$ particles that are labelled 3. We compute $n_{3,3}(s,t,n)$ in the following lemma.

Lemma 3.15. $n_{3,3}(s,t) = \binom{N-1}{s} f_{(n-2,s+t)}$.

Proof. Let M be a continuous multiline queue that is counted by $n_{3,3}(s,t,n)$, i.e.,

$$M = \begin{array}{ccccccc} a_1 & a_2 & \cdots & a_s & & & \\ b_1 & b_2 & \cdots & \cdots & b_{s+t} & & \\ c_1 & c_2 & \cdots & \cdots & \cdots & c_{n-1} & N. \\ \hline 3 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots . \end{array}$$

Here, c_1 and c_2 are not bullied by any entry in the second row. This implies that $b_1 > c_2$ and that there is no wrapping from the second to the third row. It follows that there are no restrictions on a_i and hence their values can be chosen from the set $\{1,2,\ldots,N-1\}$ in $\binom{N-1}{s}$ ways. c_1 and c_2 take the smallest two integers available after fixing a_i's. Since there are no wrappings from the second to the third row, the configurations formed by the remaining variables are in bijection with the standard Young tableaux of shape $(n-2,s+t)$ by Lemma 3.12. Therefore, they are $f_{(n-2,s+t)}$ in number. □

Remark 3.16. Following the steps in the proof of Lemma 3.15, we can give an alternate proof of (3.4.2) which is equivalent to saying $n_3(s,t,n) = \binom{N-1}{s} f_{(n-1,s+t)}$.

Thus, given (3.4.1), it suffices to find $n_{1,3}$ for the sake of our analysis. The values of $n_{1,3}(s,t,n)$ for different s and t for $n = 5, 6$ are shown in the following tables.

$s \backslash t$	1	2	3
1	9	14	14
2	126	140	0
3	770	0	0

$s \backslash t$	1	2	3	4
1	14	28	42	42
2	280	462	504	0
3	2772	3276	0	0
4	15288	0	0	0

Table 3.1: Data for $n_{1,3}(s,t,n)$ for $n = 5, 6$

By studying the values for $n_{1,3}$ for different s, t and n, we formulate the following expression for $n_{1,3}$, which we prove in Section 3.5.

Theorem 3.17. *For $s, t \geq 1$ and $n > s + t$, we have*

$$n_{1,3}(s,t,n) = \binom{N-1}{s-1} f_{(n-1,s+t)}. \tag{3.4.3}$$

Proof of Theorem 3.2. From straightforward calculations using (3.4.1), (3.4.2), Lemma 3.15 and Theorem 3.17, we have

$$n_{2,3}(s,t) = \frac{1}{N}\binom{n+2s+t}{n,\, s,\, s+t}\left(\frac{n+t}{\binom{n+s+t}{s+t}} f_{(n-1,s+t)} - \frac{n+s+t}{\binom{n+s+t}{s+t}} f_{(n-2,s+t)}\right).$$

Since $n_{2,3}(s,t,n) = \theta_{s,t} = \binom{n+2s+t}{n,s,s+t} T^{<}(s,t)$, we have

$$(n+2s+t)T^{<}(s,t) = \frac{n+t}{\binom{n+s+t}{s+t}} f_{(n-1,s+t)} - \frac{n+s+t}{\binom{n+s+t}{s+t}} f_{(n-2,s+t)}. \tag{3.4.4}$$

The proof is completed by substituting (3.4.4) in (3.2.4). First let $i + 1 < j$. Denote the terms of the right-hand side of (3.4.4) by $C(s,t)$ and $D(s,t)$ respectively. We first compute $C(i-1, j-i) - C(i, j-i-1) - C(i-1, j-i+1) + C(i, j-i) = \mathcal{C}(\text{say})$.

$$\begin{aligned}
\mathcal{C} &= \frac{f_{(n-1,j-1)}}{\binom{n+j-1}{j-1}}(((n+j-i)-(n+j-i-1)) \\
&\quad -\frac{f_{(n-1,j)}}{\binom{n+j}{j}}(((n+j-i)-(n+j-i+1)) \\
&= \frac{n}{\binom{n+j}{2}}.
\end{aligned}$$

If we define $\mathcal{D}$ similarly as $D(i-1, j-i) - D(i, j-i-1) - D(i-1, j-i+1) + D(i, j-i)$, we get $\mathcal{D} = 0$. Since, $c_{i,j}(n) = \mathcal{C} + \mathcal{D}$, we have $c_{i,j}(n) = \frac{n}{\binom{n+j}{2}}$.

Note that $T^{<}(s, 0) = 0$ by definition. Therefore when $i + 1 = j$, (3.2.4) becomes

$$c_{j-1,j} = (n+2j-3)\,T^{<}(j-2,1) - (n+2j-2)\,T^{<}(j-2,2) + (n+2j-1)\,T^{<}(j-1,1).$$

Let $\mathcal{C}' = C(j-2,1) - C(j-2,2) + C(j-1,1)$ and define $\mathcal{D}'$ similarly. Then,

$$\begin{aligned}
\mathcal{C}' &= \frac{n+1}{\binom{n+j-1}{j-1}} f_{(n-1,j-1)} - \frac{n+2}{\binom{n+j}{j}} f_{(n-1,j)} + \frac{n+1}{\binom{n+j}{j}} f_{(n-1,j)}, \\
\mathcal{D}' &= -\frac{n+j-1}{\binom{n+j-1}{j-1}} f_{(n-2,j-1)}.
\end{aligned}$$

Thus, $c_{j-1,j}(n) = \mathcal{C}' + \mathcal{D}' = \frac{ni}{\binom{n+i}{2}} + \frac{n}{\binom{n+j}{2}}$, completing the proof. □

3.5 Proof of Theorem 3.17

To compute $n_{1,3}(s, t, n)$, we need to count the number of continuous multiline queues with the following configuration. Here, $N = n + 2s + t$ while a_i, b_i and c_i are distinct integers from the set $[N]$.

$$\begin{array}{c|ccccccc}
 & a_1 & a_2 & \cdots & a_s & & & \\
 & b_1 & b_2 & \cdots & \cdots & b_{s+t} & & \\
 & c_1 & c_2 & \cdots & \cdots & \cdots & c_{n-1} & N. \\
\hline
\omega = & 1 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Since $\omega_2 = 3$, c_2 can not be bullied by any b_i. Therefore, there can be at most one wrapping from the second row to the third row. These configurations can be classified into two types:

1. there is no wrapping from the second row,

2. only b_{s+t} wraps around and bullies c_1.

Let us denote the number of continuous multiline queues from the two cases by $\alpha_{1,3}(s,t,n)$ and $\beta_{1,3}(s,t,n)$ respectively. We will enumerate them separately.

Proposition 3.18. *A continuous multiline queue of type $m_{s,t}$ where there is no wrapping from the second row projects to a word ω with $\omega_1 = 1$ and $\omega_2 = 3$ if and only if*

(1) there exists $1 \leq i \leq s$ such that $a_i \to b_1 \to c_1$, and

(2) $b_2 > c_2$.

Proof. Let M be a continuous multiline queue satisfying (1) and (2) and let ω be the projected word of M. The reverse implication is straightforward. We proceed to prove the forward implication. Note that $\omega_2 = 3$ implies $b_2 > c_2$, otherwise c_2 is bullied by either b_1 or b_2 giving $\omega_2 < 3$. Since there is no wrapping from the second row to the third row, $\omega_1 = 1$ is only possible when there exist a_i, b_j such that $a_i \to b_j \to c_1$ for some i. That is, $b_j < c_1$. Further, $b_j < c_1 < c_2 < b_2$ implies $j = 1$. □

Theorem 3.19. *Let $\alpha_{1,3}(s,t,n)$ be the number of continuous multiline queues of type $m_{s,t}$ with the largest entry N in the last row and no wrapping from the second row such that the projected word ω has $\omega_1 = 1$ and $\omega_2 = 3$. Then,*

$$\alpha_{1,3}(s,t) = \left(\binom{N-2}{s-1} - \binom{N-2}{s-3}\right) f_{(n-1,s+t)}.$$

Proof. Let the continuous multiline queues that are counted by $\alpha_{1,3}(s,t,n)$ exhibit the following configuration:

$$\begin{array}{r|ccccccc}
 & a_1 & a_2 & \cdots & a_s & & & \\
 & b_1 & b_2 & \cdots & \cdots & b_{s+t} & & \\
 & c_1 & c_2 & \cdots & \cdots & \cdots & c_{n-1} & N. \\
\hline
\omega = & 1 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Let us first assume that $a_1 < b_1$. Coupled with $b_1 < c_1$ from the proof of Proposition 3.18, we get $a_1 = 1$, and $a_1 \to b_1 \to c_1$, which gives $\omega_1 = 1$. The remaining a_i's assume increasing integer values between 2 and $N-1$ in $\binom{N-2}{s-1}$ ways. We also have $c_1 < c_2 < b_2$. Thus, b_1, c_1 and c_2 are assigned the three smallest values after eliminating integers selected by the a_i's. Because there is no wrapping in any of the rows, such configurations are in bijection with standard Young tableaux of shape $\lambda = (n-2, s+t-1)$ (see Lemma 3.12). This gives us $\binom{N-2}{s-1} f_{(n-2,s+t-1)}$ continuous multiline queues that contribute to $\alpha_{1,3}(s,t,n)$ for the case $a_1 < b_1$.

Now, let us assume that $a_1 > b_1$. Then, there exists an a_i such that $a_i \xrightarrow{W} b_1$ by Proposition 3.18(1). The inequalities $b_1 < c_1$ and $b_1 < a_1$ imply $b_1 = 1$. Instead, we first find the number of continuous multiline queues where the only constraints are $b_1 = 1, c_n = N$ and $c_2 < b_2$, with no wrapping from the second row to the third row. For these, $b_1 \to c_1$ and we get $\omega_1 \in \{1, 2\}$ and $\omega_2 = 3$. Observe that the a_i's can take any value other than 1 and N. c_1 and c_2 are then assigned the smallest two integers after eliminating 1 and the integers selected by the a_i's. Also, the configurations formed by the remaining b_i's and c_i's satisfy the following inequalities

$$\begin{array}{ccccccccccccc} & & & & b_2 & < & \dots & < & b_{s+t-j} & < & \dots & < & b_{s+t} \\ & & & & \wedge & & \cdots & & \wedge & & \cdots & & \wedge \\ c_3 & < & \dots & < & \dots & < & & & c_{n-j} & < & \dots & < & c_n, \end{array}$$

and hence they can be arranged in $f_{(n-2,s+t-1)}$ ways. The required number is given by $\binom{N-2}{s} f_{(n-2,s+t-1)}$. From this set, in order to eliminate the continuous multiline queues with $\omega_1 = 2$ and $\omega_2 = 3$, we need to subtract the number of continuous multiline queues where $b_1 = 1$, $b_2 > c_2$ with no wrapping in any row from the number $\binom{N-2}{s} f_{(n-2,s+t-1)}$. In this regard, let $c_2 = k + 3$ for some $0 \le k \le s$. Then, there are k a_i's that are smaller than c_2 and there are $k + 1$ ways to assign values to $c_1, a_1, \cdots a_k$. The remaining entries of the continuous multiline queue satisfy the following inequalities:

$$\begin{array}{ccccccccccccc} & & & & & & a_{k+1} & < & a_{k+2} & < & \cdots & < & a_s \\ & & & & & & \wedge & & \wedge & & \cdots & & \wedge \\ & & b_2 & < & \cdots & < & b_{t-k-1} & < & b_{t-k} & < & \cdots & < & b_{s+t} \\ & & \wedge & & \dots & & \wedge & & \wedge & & \cdots & & \wedge \\ c_3 & < & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & < & c_n. \end{array}$$

Such configurations are in bijection with standard Young tableaux of shape $\lambda_k = (n - 2, s + t - 1, s - k)$ which are $f_{(n-2,s+t-1,s-k)}$ in number. Thus, the number of continuous multiline queues contributing to $\alpha_{1,3}(s, t, n)$ where $a_1 > b_1$ is given by

$$\binom{N-2}{s} f_{(n-2,s+t-1)} - \sum_{k=0}^{s} (k + 1) f_{(n-2,s+t-1,s-k)}.$$

Adding this to $\binom{N-2}{s-1} f_{(n-2,s+t-1)}$ for the case $a_1 < b_1$, we get

$$\alpha_{1,3}(s,t,n) = \binom{N-2}{s-1} f_{(n-2,s+t-1)} + \binom{N-2}{s} f_{(n-2,s+t-1)} \\ - \frac{f_{(n-2,s+t-1)}}{n(s+t)} \sum_{k=0}^{s} (k+1)(t+k)(n-s+k)\binom{N-k-3}{s-k} \quad (3.5.1)$$

Summing $(k+1)(t+k)(n-s+k)\binom{N-k-3}{s-k}$ over $k=0$ to s, we get

$$(s((n+2)^2 t + 2n(n+2) + t^2 - t^3 - 4) - s^2((n+2t+1)t-6) - s^3(t+2) \\ + nt(n+t)(n+t+1))\binom{N-1}{s}.$$

Now, simplifying the right hand side of (3.5.1), we have

$$\begin{aligned} \alpha_{1,3}(s,t,n) &= \frac{(n-s-t)(n+t+2)}{(n+s+t)(n+s+t+1)}\binom{N-1}{n, s-1, s+t} \\ &= \left(\binom{N-2}{s-1} - \binom{N-2}{s-3}\right) f_{(n-1,s+t)}. \end{aligned}$$

□

Proposition 3.20. *A continuous multiline queue of type $m_{s,t}$ with N in the last row, such that there is exactly one wrapping from the second row, projects to a word ω with $\omega_1 = 1$ and $\omega_2 = 3$ if and only if*

1. *there exists $i < s$ such that $a_i \to b_{s+t-1} \to c_n$ and $a_{i+1} \to b_{s+t} \xrightarrow{W} c_1$, and*
2. *$b_1 > c_2$.*

Proof. Let M be a continuous multiline queue with N in the last row and exactly one wrapping from the second row such that the projected word ω has $\omega_1 = 1$ and $\omega_2 = 3$. If $b_1 < c_2$ in M, then c_2 is bullied by at least one b_i (either by b_1 or by the wrapping) giving $\omega_2 < 3$, a contradiction. Therefore, $b_1 > c_2$.

Then, there exists integers, say j, v such that $a_j \to b_v \xrightarrow{W} c_1$ to give $\omega_1 = 1$. If $v < s+t$, then there exists $w > v$ such that $b_w \xrightarrow{W} c_2$, once again giving a contradiction. Therefore, $v = s+t$. Further, since $a_j \to b_{s+t} \xrightarrow{W} c_1$, there exists $i < j$ and $u < s+t$ such that $a_i \to b_u \to c_n$ to give $\omega_n = 1$. Otherwise, b_{s+t} bullies $c_n = N$ and there is no wrapping from the second row to the third row. Since, $b_u \to c_n$, all of $b_{u+1}, b_{u+2}, \cdots, b_{s+t}$ wrap around to the third row. As there can be exactly one such wrapping, $u+1 = s+t$. Moreover, since $a_i \to b_{s+t-1}$, a_k wraps around to the second row for all $k > i+1$, thus proving $j = i+1$. The reverse implications are straightforward to verify. □

Recall that the number of continuous multiline queues satisfying Proposition 3.20 is counted by $\beta_{1,3}$. The values of $\beta_{1,3}(s,t,n)$ for different s and t for $n = 5, 6$ are shown in Table 3.2.

$s\backslash t$	0	1	2
2	9	14	14
3	140	154	0
4	924	0	0

$s\backslash t$	0	1	2	3
2	14	28	42	42
3	280	504	546	0
4	3276	3822	0	0
5	19110	0	0	0

Table 3.2: Data for $\beta_{1,3}(s,t,n)$ for n=5,6

By observing the above data, we formulate the following expression for β_{13}.

Theorem 3.21. $\beta_{1,3}(s,t,n) = \binom{N-1}{s-2} f_{(n-1,s+t)}$.

Remark 3.22. It is interesting to see that the techniques we have used to prove the earlier cases do not work here as there is no easy bijection which can be used to prove Theorem 3.21. We use alternative methods to give the proof of Theorem 3.21 in Section 3.5.1. For now, we independently prove Theorem 3.17 using the properties of continuous multiline queues that are counted by $\beta_{1,3}$.

We first prove that $\beta_{1,3}(s,t,n)$ satisfies a simple recurrence.

Lemma 3.23. *For $s, t \geq 2$ and $s + t < n$:*

$$\beta_{1,3}(s,t,n) = \beta_{1,3}(s-1,t+1,n) + \beta_{1,3}(s,t-1,n) + \beta_{1,3}(s,t,n-1). \tag{3.5.2}$$

Proof. Consider a continuous multiline queue M satisfying the conditions from Proposition 3.20. Let $c_2 = k + 2$ for some $k \leq s$. Then, M has the following configuration:

$$\begin{array}{cccccccccc}
a_1 & a_2 & \cdots & a_k & \cdots & a_s & & & & \\
b_1 & b_2 & \cdots & b_k & \cdots & \cdots & b_{s+t} & & & \\
c_1 & k+2 & \cdots & \cdots & \cdots & \cdots & \cdots & c_{n-1} & N. \\
\hline
1 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{array}$$

Since $b_1 > c_2$, there are only $\binom{k+1}{k} = k+1$ ways to assign values to $c_1, a_1, \ldots, a_k$ from the set $[k+1]$. Thus, for each $u \leq k$, $a_u \to b_u$. Given that the rows are strictly increasing, exactly one of the following cases is true:

1. $a_{k+1} = k + 3$:
   Here a_{k+1} bullies b_{k+1}, so a_{k+1} does not bully b_{s+t-1} or b_{s+t}. Deleting a_{k+1} and

subtracting 1 from all the values higher than $k+3$ does not affect the bully paths containing b_{s+t-1} or b_{s+t}. In the projected word ω, one of the 1's changes to a 2. So, there are $\beta_{1,3}(s-1,t+1,n)$ such continuous multiline queues.

2. $b_1 = k+3$:
   b_1 bullies c_3. Deleting b_1 and subtracting 1 from all the values higher than $k+3$ in the continuous multiline queue does not affect the bully paths containing b_{s+t-1} or b_{s+t}. In the projected word ω, one of the 2's changes to a 3. Thus, there are $\beta_{1,3}(s,t-1,n)$ such continuous multiline queues.

3. $c_3 = k+3$:
   Finally, in this case, c_3 is not bullied by any b_u because $b_1 > c_3$ and there is exactly one wrapping from the second to the third row. Here, deleting c_3 and subtracting 1 from all the values higher than $k+3$ does not affect any bully path and the length of the resulting projected is reduced by one. There are $\beta_{1,3}(s,t,n-1)$ such continuous multiline queues.

Therefore, $\beta_{1,3}(s,t,n)$ is obtained by adding the numbers in each of the above cases. $\square$

We can verify the equation

$$\alpha_{1,3}(s,t,n) = \alpha_{1,3}(s-1,t+1,n) + \alpha_{1,3}(s,t-1,n) + \alpha_{1,3}(s,t,n-1),$$

for $s,t \geq 2$ and $n > s+t$ by plugging the value of $\alpha_{1,3}(s,t,n)$ from Theorem 3.19. This along with Lemma 3.23 gives the recurrence relation

$$n_{1,3}(s,t,n) = n_{1,3}(s-1,t+1,n) + n_{1,3}(s,t-1,n) + n_{1,3}(s,t,n-1),$$

for $s,t \geq 2$ and $n > s+t$. Let $P(s,t,n)$ denote the product $\binom{N-1}{s-1} f_{n-1,s+t}$ from Theorem 3.17. We have

$$P(s,t,n) = P(s-1,t+1,n) + P(s,t-1,n) + P(s,t,n-1),$$

using Pascal's rule and the hook length recurrence relation $f_{(a,b)} = f_{(a-1,b)} + f_{(a,b-1)}$ where $a \geq b > 1$ (see Remark 2.9). As a result, $P(s,t,n)$ satisfies the same recurrence relation as $n_{1,3}(s,t,n)$. Moreover, $n_{1,3}(s,t,s+t) = 0$ as there is no 3 in the projected word. We also have

$$n_{1,3}(1,t,n) = \alpha_{1,3}(1,t,n) = f_{(n-1,t+1)},$$

as $\beta_{1,3}(1,t,n) = 0$ for all t,n. This holds because for a continuous multiline queue with exactly one wrapping from the second row to the third row which projects to a word beginning with $(2,3)$, we need s to be greater than 1, by Proposition 3.20. Thus, proving the initial condition $n_{1,3}(s,1,n) = P(s,1,n)$ completes the proof of Theorem 3.17. To that end, we have the following result.

Proposition 3.24. *For s,n such that $1 < s+1 < n$, we have*

$$n_{1,3}(s,1,n) = \binom{n+2s}{s-1} f_{(n-1,s+1)}. \tag{3.5.3}$$

Proof. Recall that $m_{s,t} = (s,t,n-s-t)$. Also, $n_{x,y}(s,t,n)$ counts the number of continuous multiline queues M of type $m_{s,t}$ with N in the last row, such that M projects to a word ω where $\omega_1 = x$ and $\omega_2 = y$.

$$M = \begin{array}{ccccccc} a_1 & a_2 & \cdots & a_s & & & \\ b_1 & b_2 & \cdots & \cdots & b_{s+t} & & \\ c_1 & c_2 & \cdots & \cdots & \cdots & c_{n-1} & N. \\ \hline x & y & \cdots & \cdots & \cdots & \omega_{n-1} & \omega_n \end{array}$$

Let $\mu_{x,y}$ be the probability that a particle labelled x is immediately followed by a particle labelled y in a continuous TASEP of type $m_{s,t}$, with x followed by y. Then by Lemma 3.11,

$$\mu_{x,y} = \frac{n+2s+t}{\binom{n+2s+t}{s,s+t,n}} n_{x,y}.$$

Consider M', a continuous multiline queue of type $m_u = (u, n-u)$ such that the largest entry $N' = n+u$ is in the last row. Recall from Section 3.2.3, that $\delta_{c,d}(u,n)$ counts the number of continuous multiline queues M' with N' in the last row, such that M' projects to a word ω' where $\omega'_1 = c$ and $\omega'_2 = d$.

$$M' = \begin{array}{cccccc} a_1 & a_2 & \cdots & a_u & & \\ b_1 & b_2 & \cdots & \cdots & b_{n-1} & N' \\ \hline c & d & \cdots & \cdots & \omega'_{n-1} & \omega'_n \end{array}.$$

From (3.2.11), we know that $\delta_{2,2}(u,n) = f_{(n-2,u)}$. Let $\epsilon_{c,d}(n)$ denote the probability that a particle labelled c is immediately followed by one labelled d in a continuous two-species TASEP of type m_u. Then, again by Lemma 3.11,

$$\epsilon_{c,d} = \frac{n+u}{\binom{n+u}{u,n}} \delta_{c,d}.$$

We can define a lumping (or a colouring) for the continuous three-species TASEP of type $m_{s,t}$ to a continuous two-species TASEP as follows. Let $\Omega_{s,t}$ and Ω_s be the set of labelled words on a ring of type $m_{s,t}$ and $m_s = (s, n-s)$ respectively. Let $f : \Omega_{s,t} \to \Omega_s$ be a map defined as follows:

$$f(\omega_1, \dots, \omega_n) = (f(\omega_1), \dots, f(\omega_n)),$$

where

$$f(i) = \begin{cases} 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, 3. \end{cases}$$

$x \backslash y$	1	2	3
1	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{1,3}$
2	$\mu_{2,1}$	$\mu_{2,2}$	$\mu_{2,3}$
3	$\mu_{3,1}$	$\mu_{3,2}$	$\mu_{3,3}$

Table 3.3: The table contains the correlations of two adjacent particles in $\Omega_{s,t}$. The table is divided into four parts, and entries of the yellow, green, red and blue sections contribute to the correlations $\epsilon_{1,1}, \epsilon_{1,2}, \epsilon_{2,1}$ and $\epsilon_{2,2}$ respectively in Ω_s.

By lumping the Markov process, we have

$$\epsilon_{2,2} = \{\mu_{2,2} + \mu_{3,2} + \mu_{2,3} + \mu_{3,3}\}.$$

That is,

$$\frac{n+s}{\binom{n+s}{s}}\delta_{2,2} = \frac{n+2s+t}{\binom{n+2s+t}{n,s,s+t}}\{n_{2,2} + n_{3,2} + n_{2,3} + n_{3,3}\}. \tag{3.5.4}$$

Let $t = 1$. Then, $n_{2,2} = 0$ because there is only one particle with label 2 in the three-species continuous TASEP. Note that $n_{3,2}(s, t, n) = \tau_{s,t}$ from Section 3.3. Thus, substituting $t = 1$ in (3.3.4), we obtain

$$\begin{aligned} \frac{n+2s+1}{\binom{n+2s+1}{n,s,s+1}}\tau_{s,t} &= \frac{n(s+1)}{(n+s)(n+s+1)}\left(-1 + \frac{s}{n} + \frac{(n+s)(n^2+n-s(s+1)-2)}{(n+s-1)(s+1)(n+s+t)}\right) \\ &= \frac{n(s+1)}{(n+s)(n+s+1)} \times \frac{-n(s+1)(n+s-1) + s(s^2-1) + (n^3+n^2-2n)}{n(n+s-1)(s+1)} \\ &= \frac{(n+s)(n-s-1)(n-s+1)}{(n+s)(n+s-1)(n+s+1)} \end{aligned}$$

Therefore,

$$\frac{n+2s+1}{\binom{n+2s+1}{n,s,s+1}} n_{3,2}(s,1,n) = \frac{(n-s-1)(n-s+1)}{(n+s-1)(n+s+1)}. \tag{3.5.5}$$

We also have

$$\{n_{1,3}+n_{2,3}+n_{3,3}\}(s,1,n) = \frac{n-s-1}{n+2s+1}\binom{n+2s+1}{n,s,s+1}, \tag{3.5.6}$$

by substituting $t=1$ in (3.4.1). Solving (3.5.6) for $n_{1,3}$ using (3.5.4) and (3.5.5) gives

$$\begin{aligned} n_{1,3}(s,1,n) &= \frac{s(n-s-1)}{(n+s+1)(n+2s+1)}\binom{n+2s+1}{n,s,s+1} \\ &= \binom{n+2s}{s-1} f_{(n-1,s+1)}, \end{aligned}$$

proving the result. ☐

3.5.1 Proof of Theorem 3.21

We can now prove Theorem 3.21 directly using Theorem 3.17 as follows.

Proof of Theorem 3.21.

$$\begin{aligned} \beta_{1,3}(s,t,n) &= n_{1,3}(s,t,n) - \alpha_{1,3}(s,t,n) \\ &= \left(\binom{N-1}{s-1} - \binom{N-2}{s-1} + \binom{N-2}{s-3}\right) f_{(n-1,s+t)} \\ &= \binom{N-1}{s-2} f_{(n-1,s+t)}. \end{aligned}$$

☐

In addition, we describe the developments made towards a direct proof of Theorem 3.21 using the first principles in this section. Recall the recurrence relation (3.5.2). The equation holds true for $s,t \geq 2$ and $s+t \leq n$. It is easy to verify that the product $\binom{N-1}{s-2} f_{(n-1,s+t)}$ satisfies the same recurrence relation as $\beta_{1,3}(s,t,n)$. Thus, it is sufficient to show that the initial conditions are the same for both quantities. The conditions are:

$$\begin{aligned} \beta_{1,3}(1,t,n) &= 0, \\ \beta_{1,3}(s,t,s+t) &= 0, \\ \beta_{1,3}(s,1,n) &= \binom{N-1}{s-2} f_{(n-1,s+1)}. \end{aligned}$$

The first two initial conditions are straightforward. We now provide a formula for $\beta_{1,3}$ for $t = 1$.

Lemma 3.25.

$$\beta_{1,3}(s,1,n) = \sum_{\ell=2}^{s} \gamma^{\ell}(s,n),$$

where $\gamma^{\ell}(s,n)$ is the number of continuous multiline queues of type $m_{s,1}$ with $c_2 = \ell, c_n = N$ and $b_1 > c_2$, such that the projected words have 1 and 3 in the first two places respectively.

Proof. Consider a continuous multiline queue M of type $m_{s,1}$ that is counted by $\beta_{1,3}(s,1,n)$:

$$M = \begin{array}{cccccccc} a_1 & a_2 & \cdots & \cdots & a_s & & & \\ b_1 & b_2 & \cdots & \cdots & b_s & b_{s+1} & & \\ c_1 & c_2 & \cdots & \cdots & \cdots & \cdots & c_{n-1} & N. \\ \hline 1 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{array}$$

Let c_2 be equal to ℓ which is greater than 1. According to Proposition 3.20, $c_2 < b_1$. Therefore, the values $\{1,\ldots,\ell-1\}$ are assigned to $c_1, a_1, \ldots, a_{\ell-2}$. Since there exists i such that $a_i \to b_s$ and $a_{i+1} \to b_{s+1}$, ℓ ranges from $\{2,\ldots,s\}$. The set of all continuous multiline queues counted by $\beta_{1,3}(s,1,n)$ can be divided into smaller sets depending on the value of ℓ. We denote the number of such continuous multiline queues that have $c_2 = \ell$ as $\gamma^{l}(s,n)$. By adding over all possible values of ℓ, we obtain the required expression. $\square$

Next, we prove a formula for γ^{ℓ} from the first principles. First, consider a skew shape λ/μ, where λ and μ are partitions such that $\mu \subseteq \lambda$ in containment order. Recall from Section 2.2, that the number of standard Young tableaux of a skew shape λ/μ is given by $f_{\lambda/\mu}$ and it can be computed using (2.2.4).

Theorem 3.26.

$$\begin{aligned} \gamma^{\ell}(s,n) \quad = \quad & (\ell-1) \sum_{i=\ell-1}^{s-1} \sum_{j=\ell-2}^{i-2} \sum_{k=2}^{n-s+i-1} f_{(n-s+i-3,i-2,j-\ell+2)/(n-s+i-k-1)} \times \\ & \left(f_{(n+i-k-j,s-j,s-j)/(i-j+1,i-j-1)} - f_{(n+i-k-j-1,s-j,s-j)/(i-j,i-j-1)} \right). \end{aligned} \tag{3.5.7}$$

Proof. Let $c_2 = \ell$. Then, $\gamma^{\ell}(s,n)$ counts the number of continuous multiline queues M

with the following configuration.

$$M = \begin{array}{cccccc} a_1 & a_2 & \cdots & a_s & & \\ b_1 & b_2 & \cdots & b_s & b_{s+1} & \\ c_1 & \ell & \cdots & \cdots & \cdots & N, \\ \hline 1 & 3 & \cdots & \cdots & \cdots & 1 \end{array}$$

where $b_1 > c_2$, such that $c_1, a_1, \dots a_{\ell-2} \in [\ell-1]$. We can choose c_1 in $(\ell-1)$ different ways, and the choice determines the values of a_1 to $a_{\ell-2}$. These values must satisfy $a_u < b_u$ for $1 \leq u \leq \ell-2$, implying that $a_u \to b_u$ for each u.

Since M is of type $(s, 1, n-s-1)$, there is exactly one b_i which is not bullied by any a_j in the first iteration of the bully path process. Here, i can range from $\ell-1$ to $s-1$. For a fixed i, we can split the set of entries of a continuous multiline queue into two sets based on their relation with b_i. Let j and k be the largest integers such that $a_j < b_i$ and $c_k < b_i$ respectively. The value of j lies between $\ell-2$ and $i-1$ by the choice of i and j. And $n-k > s-i$ ensures that there is no more than one wrapping from the second row to the third row which implies that k lies between 0 and $n-s+i-1$. For the continuous multiline queues with at most one wrapping from the second row to the third row, the following inequalities hold for fixed i, j and k according to Proposition 3.20.

$$\begin{array}{ccccccccccccccccc}
 & & a_{\ell-1} & < & \dots & < & a_j & & & & a_{j+1} & < & \dots & \dots & < & \dots & < a_s \\
 & & \wedge & & \cdots & & \wedge & & & & \wedge & & \cdots & \wedge & & \wedge & \\
 & b_1 < & \dots & < & \dots & < & b_{i-1} & < b_i < & & & b_{i+1} & < & \dots & b_s & < & b_{s+1} & \\
 & \wedge & \cdots & & \wedge & & & & & & \wedge & & \cdots & \wedge & & & \\
c_3 < & \dots & \dots & < & c_k & & & & c_{k+1} & < & \cdots & < & \cdots & c_n. & & &
\end{array}$$

These arrangements are counted by the product of hooks length formulas of appropriate skew shapes, i.e. by,

$$f_{\lambda_1/\mu_1} . f_{\lambda_2/\mu_2}, \tag{3.5.8}$$

where $\lambda_1/\mu_1 = (n-s+i-3, i-1, j-\ell)/(n-s+i-k-1)$, $\lambda_2/\mu_2 = (n+i-k-j, s-j, s-j)/(i-j+1, i-j-1)$.

To obtain the required number of continuous multiline queues with exactly one wrapping from the second row to the third row, we have to remove the continuous multiline queues with no wrapping from the second row to the third row from the above set. These multiline queues are determined by the inequalities:

$$
\begin{array}{cccccccccccccccc}
 & & a_{\ell-1} & < & \cdots & < & a_j & & & & a_{j+1} & < & \cdots & < & \cdots & < a_s \\
 & & \wedge & & \cdots & & \wedge & & & & \wedge & & \cdots & & \wedge & \\
 & b_1 < & \cdots & < & \cdots & < & b_{i-1} & < b_i < & & & b_{i+1} & < & \cdots & < & b_{s+1} & \\
 & \wedge & \cdots & & \wedge & & & & & & \wedge & & \cdots & & \wedge & \\
c_3 < & \cdots & \cdots & < & c_k & & & & c_{k+1} & < & \cdots & < & \cdots & < & c_n. &
\end{array}
$$

The number of these arrangements is

$$
f_{\lambda_1/\mu_1} . f_{\lambda_3/\mu_3}, \tag{3.5.9}
$$

where $\lambda_3/\mu_3 = (n+i-k-j-1, s-j, s-j)/(i-j, i-j-1)$.

Summing the difference of two products in (3.5.8) and (3.5.9) over all possible values of i, j, k, and multiplying the sum with $\ell - 1$ for each choice of c_1, we prove the result. $\square$

Remark 3.27. Unfortunately, we have not been able to find a closed-form expression of the sum on the right-hand side of (3.5.7). However, by doing extensive numerical checks, we have a conjecture formulating $\gamma^\ell(s,n)$.

Conjecture 3.28. *We have,*

$$
\gamma^\ell(s,n) = (\ell-1)\binom{n+2s-\ell}{s-\ell} f_{(n-1,s+1)}.
$$

Assuming Conjecture 3.28, we can immediately prove by summing over ℓ that

$$
\beta_{1,3}(s,1,n) = \binom{n+2s}{s-2} f_{(n-1,s+1)},
$$

. This gives an alternative proof of Theorem 3.21 given Conjecture 3.28

Next, we demonstrate an approach to prove Conjecture 3.28. First, consider the set $\mathcal{H}$ of continuous multiline queues of type $m_{s,1}$ that have $c_1 = 1$, $c_2 = 2$ and $c_n = N$ where $N = n+2s+1$ is the largest entry. Let $\rho_{c,d}(s,n)$ denote the number of continuous multiline queues in $\mathcal{H}$ that project to a word starting with c, d. That is, a continuous multiline queue in $\mathcal{H}$ that contributes to $\rho_{c,d}(s,n)$ has the following structure:

$$
\begin{array}{ccccccc}
a_1 & a_2 & \cdots & \cdots & a_s & & \\
b_1 & b_2 & \cdots & \cdots & b_s & b_{s+1} & \\
1 & 2 & \cdots & \cdots & \cdots & \cdots & N. \\
\hline
c & d & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

Similarly, let $\rho_d(s,n)$ denote the number of continuous multiline queues in $\mathcal{H}$ that project to a word with d at the second position. Then for s, n we have,

$$\rho_{1,3}(s,n) + \rho_{2,3}(s,n) + \rho_{3,3}(s,n) = \rho_3(s,n). \tag{3.5.10}$$

Note that $\rho_{1,3}(s,n) = \gamma^2(s,n)$. Moreover, $\rho_{3,3}(s,n) = \binom{N-3}{s} f_{(n-2,s+1)}$ following the same lines of arguments as in Lemma 3.15. For ρ_3, consider a continuous multiline queue of the form

$$\begin{array}{ccccccc} a_1 & a_2 & \cdots & \cdots & a_s & & \\ b_1 & b_2 & \cdots & \cdots & b_s & b_{s+1} & \\ 1 & 2 & \cdots & \cdots & \cdots & \cdots & N. \\ \hline \cdot & 3 & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

To count such configurations, note that there is no restriction on the a_i's and they can be assigned any values from the set $\{3, \dots, N-1\}$. There can at most be one wrapping from the second row to the third row, thus the remaining entries satisfy the following inequalities:

$$\begin{array}{ccccccccccccc} & & b_1 & < & \dots & < & b_{s-1} & < & b_s & < & b_{s+1} \\ & & \wedge & & \cdots & & \wedge & & \wedge & & \\ c_3 & < & \dots & < & \dots & < & c_{n-1} & < & N. & & \end{array}$$

The arrangements are in bijection with standard Young tableaux of skew shape $(n-1, s+1)/(2)$. Therefore,

$$\rho_3(s,n) = \binom{N-3}{s} f_{(n-1,s+1)/(2)}. \tag{3.5.11}$$

Hence, it is enough to compute $\rho_{2,3}(s,n)$ in order to find $\gamma^2(s,n)$ using (3.5.10), which in turn gives $\beta_{1,3}(s,1,n)$. The values of $\rho_{2,3}(s,n)$ for different s and n are shown in Table 3.4. By observing Table 3.4, we conjecture the following formula for $\rho_{2,3}(s,n)$.

$s \backslash n$	3	4	5	6
1	3	4	5	6
2	0	40	70	112
3	0	0	630	1260
4	0	0	0	11088

Table 3.4: $\rho_{2,3}(s,n)$ for different values of s and n

Conjecture 3.29. *We have*

$$\rho_{2,3}(s,n) = \binom{N-3}{n-1} f_{(s,s)}.$$

Assuming Conjecture 3.29, we have $\rho_{1,3} = \gamma^2(s,n) = \binom{N-3}{s-2} f_{(n-1,s+1)}$, by (3.5.10) and (3.5.11).

Since $\binom{N-3}{n-1} = \frac{N-3}{n-1}\binom{N-4}{n-2}$, proving Conjecture 3.29 is equivalent to proving the following recurrence.

Conjecture 3.30. *We have*

$$(n-1)\rho_{2,3}(s,n) = (n+2s-2)\rho_{2,3}(s,n-1).$$

We now give a formula for $\rho_{2,3}(s,n)$ in terms of hook length formula for skew shapes using the first principles. The following triple sum formula gives a formula for $\rho_{2,3}(s,n)$.

Theorem 3.31. *Let $\rho_{2,3}(s,n)$ denote the number of continuous multiline queues with $c_1 = 1, c_2 = 2$ and $c_n = N$ that project to words beginning with $(2,3)$. We have*

$$\rho_{2,3}(s,n) = \sum_{i=1}^{s+1}\sum_{j=0}^{i-1}\sum_{k=0}^{i-1} f_{(s-j+k,s-i+k+1,s-i+k)/(k,k)} \times \\ \{f_{(n+i-s-3,i-1,j)/(k)} - f_{(n+s-i-4,i-1,j)/(k-1)}\}. \quad (3.5.12)$$

Proof. There exists a unique $i \in [s+1]$ such that a_j does not bully b_i for any j. Also $b_i \to c_1$ by wrapping around from the second to the third row. We already have $c_1 = 1, c_2 = 2$ and $c_n = N$. We can split the set of remaining entries from each of the three rows into two sets depending on their relation with b_i. Let j and d be the largest integers such that $a_j < b_i$ and $c_d < b_i$. Here, j lies between 0 and $i-1$. Since there is no more than one wrapping from the second row to the third row, we have $n-d > s+1-i$ which implies that $d < n-s+i-1$. Further, $d > n-s-1$ as $b_i > b_1 > c_2$. For fixed i,j,d, following inequalities hold:

$$\begin{array}{ccccccccccccccccccc}
 & & & & a_1 & < & \cdots & < & a_j & & & & a_{j+1} & < & \cdots & < & \cdots & < a_s \\
 & & & & \wedge & & \cdots & & \wedge & & & & \wedge & & \cdots & & \wedge & \\
 & & b_1 & < & \cdots & < & \cdots & < & b_{i-1} & & < b_i < & & b_{i+1} & < & \cdots & < & b_{s+1} & \\
 & & \wedge & & \wedge & & & & & & & & \wedge & & \cdots & & \wedge & \\
c_3 & < & \cdots & < & c_d & & & & & & & c_{d+1} & < & \cdots & < & \cdots & < & N
\end{array}$$

Let k be the number of extra entries of the third row that are sticking out in the third row of the skew shape on the right of b_i, i.e., $k = (n-d) - (s-i+1)$. The range of k as inferred from the range of d is $0 \leq k < i$. These arrangements are counted by the product of the numbers of standard Young tableaux for the appropriate skew shapes, i.e. by,

$$f_{\lambda_1/\mu_1}.f_{\lambda_2/\mu_2}, \tag{3.5.13}$$

where $\lambda_1/\mu_1 = (n+i-s-3, i-1, j)/k$ and $\lambda_2/\mu_2 = (s-j+k, s-i+k+1, s-i+k)/(k,k)$.

From these arrangements, for $k > 1$, we have to remove the arrangements which do not result in b_i wrapping from the second row to the third row. These are obtained by shifting the bottom row of inequalities in the skew shape on the left of b_i by 1 position towards the right. The resulting inequalities are as follows:

$$\begin{array}{ccccccccccccccccccc}
 & & & & a_1 & < & \cdots & < & a_j & & & a_{j+1} & < & \cdots & < & \cdots & < a_s \\
 & & & & \wedge & & \cdots & & \wedge & & & \wedge & & \cdots & & \wedge & \\
 & & b_1 & < & \cdots & < & \cdots & < & b_{i-1} & < b_i < & & b_{i+1} & < & \cdots & < & b_{s+1} & \\
 & & \wedge & & \ldots & & \wedge & & & & & \wedge & & \ldots & & \wedge & \\
c_3 & < & \cdots & < & \cdots & < & c_d & & & c_{d+1} & < & \cdots & < & \cdots & < & N &
\end{array}$$

The number of these multiline queues is

$$f_{\lambda_3/\mu_3}.f_{\lambda_2/\mu_2}, \tag{3.5.14}$$

where $\lambda_3/\mu_3 = (n+i-s-4, i-1, j)/(k-1)$.

Summing the difference of two products in (3.5.13) and (3.5.14) over all possible values of i, j, k, we get Theorem 3.31. $\square$

Chapter 4

Correlations in the multispecies PASEP on a ring

4.1 Introduction

The focus of this chapter is the multispecies partially asymmetric simple exclusion process or PASEP. We are interested in calculating the correlations of the first two adjacent particles in a multispecies PASEP on a finite ring. Martin [80] studied the stationary distribution of the multispecies PASEP on a finite ring and developed a method to sample exactly from the stationary distribution. He further gave results on the common denominator of the stationary distribution and a few asymptotic results for large systems. This analysis was done by using queuing systems which are constructed recursively and can be seen as multiline diagrams or multiline queues.

This chapter proves a result of the two-point correlations in the stationary distribution of the multispecies PASEP on a finite ring. To carry out this investigation, we use the multiline process described in [80] and the procedure of lumping [74] to transform the study of the stationary distribution of the multispecies PASEP into that of the stationary distribution of the multiline process. This is very similar to the technique we have already used in Chapter 3. We define bully paths on the multiline diagrams to project them to a word and assign weights to each such projection. The probability of each projection is defined in terms of the weights assigned to them.

4.2 Background and Results

To define the multispecies PASEP model, we consider a ring with a finite number of sites; some of which are occupied by n different types of particles. The unoccupied sites or *holes* are then assigned the label $n+1$ and are treated as particles with the highest label. In a PASEP model, particles move preferentially in one direction, conventionally from left to right. Let $q \in [0,1]$ be a parameter associated with this PASEP, known as the *asymmetry parameter* of the system which signifies the rate at which particles flow in the non-preferred direction in the asymmetric simple exclusion process. $q = 1$ is the limiting case, and the limiting model is known as a *symmetric simple exclusion process* or an SSEP. We will now give a formal description.

Let $m = (m_1, \ldots, m_{n+1})$ be a tuple of nonnegative integers and let $N = \sum m_i$. A multispecies PASEP of type m is a Markov process which is defined on a ring with N sites. For each $i \in [n]$, there are m_i particles with label i that occupy the sites of the ring. There are also m_{n+1} holes. Each site can accommodate at most one particle. Let the state-space of the system be denoted by Λ_m and the states are given by the cyclic words $\omega = (\omega_k : k \in [N])$, where $\omega_k \in [n+1]$ is the label of the site k. The dynamics of the process are as follows. Each particle carries an exponential clock which rings with rate 1, and the particle exchanges position with the particle on the right whenever the clock rings. Let the particle on the left and the right be labelled i and j respectively. The transitions happen with the following rates.

$$ij \to ji \text{ with rate } \begin{cases} 1, & \text{if } i > j, \\ q, & \text{if } i < j. \end{cases}$$

In this chapter, we are interested in studying the correlations of the two particles at first and the second sites of the ring for a PASEP of type $\langle 1^{\mathrm{n}} \rangle = (1, \ldots, 1)$ on a ring with n sites. Let $c^q_{i,j}(n)$ denote the probability that particles with labels i and j are in the first and the second positions (respectively) of the ring $\mathbb{Z}_n$ in the stationary distribution. Note that for $q = 0$, the process becomes a totally asymmetric simple exclusion process or a TASEP on a finite ring for which the analysis has already been done by Ayyer and Linusson [20]. Let $c^0_{i,j}(n)$ denote the probability that the first two sites of the ring are occupied by particles labelled i and j respectively $q = 0$. This is formulated in the following theorem.

Theorem 4.1. *[20, Theorem 4.2] We have for $i, j \in [n]$,*

$$c^0_{i,j} = \begin{cases} \frac{i-j}{n\binom{n}{2}}, & \text{if } i > j, \\ \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)}, & \text{if } i = j-1, \\ \frac{1}{n^2}, & \text{if } i < j-1. \end{cases}$$

We generalise Theorem 4.1 for arbitrary $q \in [0,1)$ and prove the following main theorem regarding the two-point correlations in this chapter.

Theorem 4.2. *Let $c^q_{i,j}(n)$ be the probability that particles labelled i and j are in the first and the second positions respectively in a PASEP of type $\langle 1^{\mathrm{n}} \rangle = (1, \ldots, 1)$. For $1 \le j < i \le n$, we have*

$$c^q_{i,j}(n) = c^0_{i,j}(n) - \frac{(i-j+2)(j-1)(n-i)q[i-j+1]_q}{n^2(n-1)[i-j+2]_q} - \frac{j(i-j)(n-i+1)q[i-j-1]_q}{n^2(n-1)[i-j]_q} + \frac{(i-j+1)(2j(n-i)+i+j-n-1)q[i-j]_q}{n^2(n-1)[i-j+1]_q}, \quad (4.2.1)$$

and for $1 \le i < j \le n$, we have

$$c^q_{i,j}(n) = c^0_{i,j}(n) + \frac{i(j-i)(n-j+1)q^{(j-i-1)}}{n^2(n-1)[j-i]_q} + \frac{(i-1)(j-i+2)(n-j)q^{(j-i+1)}}{n^2(n-1)[j-i+2]_q} - \frac{(j-i+1)(2i(n-j)+i+j-n-1)q^{(i-j)}}{n^2(n-1)[j-i+1]_q}, \quad (4.2.2)$$

where $[k]_q = 1 + q + \cdots + q^{k-1}$ is the q-analog of an integer $k > 0$.

Remark 4.3. Note that setting $q = 0$ in (4.2.1) and (4.2.2) gives $c^q_{i,j}(n) = c^0_{i,j}(n)$.

We prove this result using Martin's multiline process [80] and lumping defined in Section 2.1.2. We begin by defining some notation. Consider the tuple $m = (m_1, \ldots, m_{n+1})$ such that $N = m_1 + \cdots + m_{n+1}$. To construct a multiline queue or an MLQ of type m, take a cylinder of n rings numbered from top to bottom; each having N sites. For every $k \in [n]$, $S_k = m_1 + \cdots + m_k$ sites on the k^{th} ring are occupied by particles and there is no other constraint. We denote an occupied site with a $\bullet$ and an unoccupied site or a hole with a $\circ$. Refer to Figure 4.1 for an example of a multiline queue of type $(2, 1, 2, 2, 6)$. Let Ω_m be the set of all multiline queues of type m. Since the choice of sites to be occupied on different lines is independent, it is easy to see that the number of multiline queues in Ω_m is $\binom{N}{S_1}\binom{N}{S_2}\cdots\binom{N}{S_n}$.

Each multiline queue can be projected to many possible words of type m using *q-bully path* algorithm from [80] which is a generalisation of the bully path algorithm given in Section 3.2.1. It is a recursive process where at each step we pick a topmost row where a particle is available and connect an occupied site in it to an available particle in the next row and so on till a particle in the last row is linked. This is repeated until every $\bullet$ in the first $n-1$ rows is linked with a $\bullet$ in a row next to it. We call each connection between two particles on adjacent rows a *link* (denoted by $\rightarrow$) and each such link is assigned a

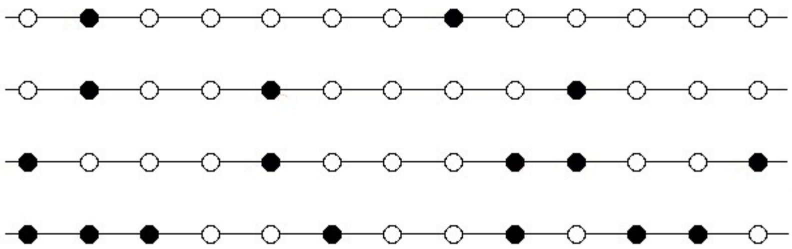


Figure 4.1: A multiline queue of type $(2, 1, 2, 2, 6)$

weight according to the number of particles available for bullying at that step. Corteel, Mandelshtam and Williams [32] use the term *pairing* to denote these links. A *linked multiline queue* or an LMLQ is defined as a multiline queue along with a maximal given set of bully links. See Figure 4.2 for an example of a linked multiline queue.

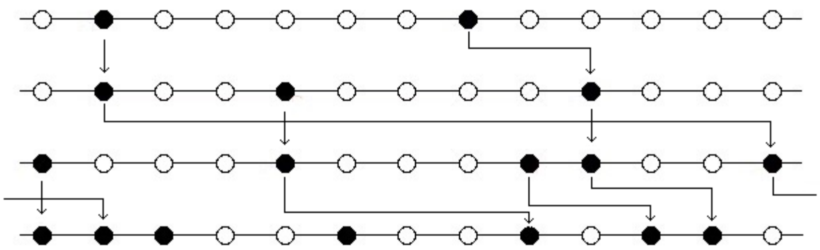


Figure 4.2: A linked multiline queue of type $(2, 1, 2, 2)$

Each linked multiline queue is associated with a word and a weight. The weight of an LMLQ is given by the product of the weights of all the links in it. We now give a description of the q-bully path algorithm first for the case $n = 2$, and later for general n. Let $m = (m_1, m_2, m_3)$, i.e., each site of the multispecies PASEP of type m is either a hole or has a particle of type 1 or of type 2. Consider a multiline queue M of type m.

(1) Choose an occupied site a in the first row of M. If there is a particle at site a in the second row as well, we construct a straight link in the a^{th} column of M and assign to it a weight 1. We call this link a “trivial” link.

(2) Otherwise, let there be t available particles in the second row at sites $b_1, \ldots, b_t$. We reorder these t particles in increasing order of the values $(b_j - a) \mod N$. The particle at a can link to any of the t particles giving us many possible LMLQs. If it

is linked to the particle at site b_i, then the link $a \to b_i$ has weight

$$\frac{q^{i-1}}{[t]_q}.$$

The particle at site b_i is now unavailable for further linking. See Figure 4.3 for examples of a link. Repeat this process by choosing particles in the first row in an arbitrary order and linking it to a particle in the second row. In Example 4.4 below, we proceed from left to right.

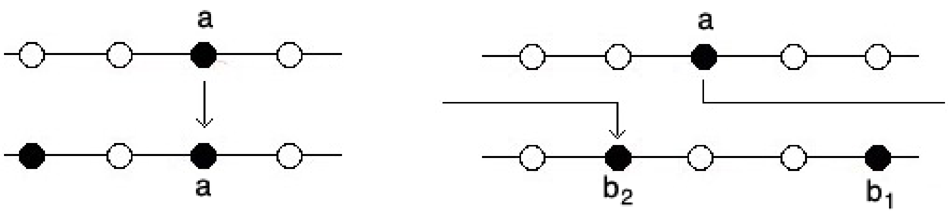


Figure 4.3: Examples of a trivial and a non-trivial link. The weight of the trivial link on the left is 1, and the weight of the non-trivial link on the right is $\frac{q}{[2]_q}$.

(3) Label all the linked particles in the second row as type 1, the particles that are not linked as type 2 and the unoccupied sites as type 3. This algorithm thus creates a linked multiline queue of M. The associated word $\omega = (\omega_i : i \in [3])$, where ω_i is the label of the site i in the second row of the multiline queue, is called the *projected word* of the LMLQ.

The weight of a linked multiline queue is the product of all the link weights. The probability of a word ω of type m in Λ_m is proportional to the sum of weights of all the LMLQs that project to ω. Next, we see an example of the q-bully path algorithm with the help of a few different linked multiline queues of the same multiline queue.

Example 4.4. Let $m = (3, 2, 3)$. Let M be a multiline queue of type m, i.e.,

$$M = \begin{matrix} \circ & \circ & \bullet & \circ & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \circ & \bullet & \circ & \bullet & \bullet \end{matrix}.$$

Figure 4.4 illustrates a few different linked multiline queues of M each projecting to a word in Λ_m. Note that two or more different LMLQs can project to the same word; evident from the first two examples in Figure 4.4.

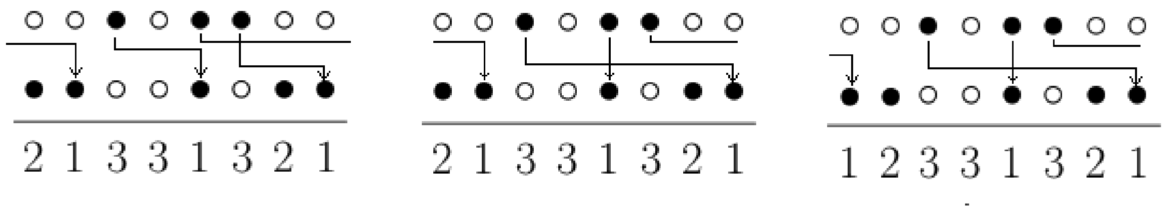


Figure 4.4: Linked multiline queues of an MLQ of type (3,2,3). The weights of the linked MLQs from left to right are $\frac{1\cdot q^3\cdot q}{[5][4][3]}$, $\frac{q^2\cdot q^2}{[5][3]}$ and $\frac{q^2\cdot q}{[5][3]}$ respectively.

The q-bully path algorithm can be seen in terms of a queueing process. The indices with $\bullet$ in the first row can be interpreted as "arrival times" in a system of queues, and the indices with $\bullet$ in the second row are "service times". For each arrival time, the algorithm assigns a different "departure time" from the available service times. The sites in the second row are thus classified into departure times, times of unused service or times of no service. These are labelled as $1, 2$ and 3 respectively in the projected word.

Now we define the algorithm for $n > 2$ recursively.

1. Let $m = (m_1, \ldots, m_{n+1})$ be such that $N = \sum m_i$. Consider M, a multiline queue of type m. Link all the particles in the first row to m_1 particles in the second row by following the q-bully path algorithm for $n = 2$ described earlier. Label the sites in the second row $1, 2$ or 3 accordingly.

2. For $k > 1$, let us assume that all the particles in the i^{th} row have been linked to an occupied site in the $(i+1)^{st}$ for all $i < k$ such that the sites in the k^{th} row are thus labelled 1 to $k+1$ depending on the links from $(k-1)^{th}$ row. Repeat Steps (1) and (2) of the algorithm for case $n = 2$ for all the particles in the k^{th} row in the increasing order of their types. Label the linked particles of $(k+1)^{th}$ row by the index of the site on the k^{th} row to which it is linked. The unlinked particles in $(k+1)^{th}$ row are labelled $k+1$ and the holes are labelled $k+2$. Similar to the Step (2) for case $n = 2$, a weight is assigned to the links according to the number of available particles.

3. Repeat the above step for all $k \leq n$. We now have a linked multiline queue. The weight of the LMLQ is given by the product of the weights of all the links in it. The labels of all the sites in n^{th} row generate a word $\omega = (\omega_i : i \in [n+1])$. Refer to Figure 4.5 for an example of q-bully path projection on a multiline queue with 4 rows; with the product of link weights of each row mentioned towards its right.

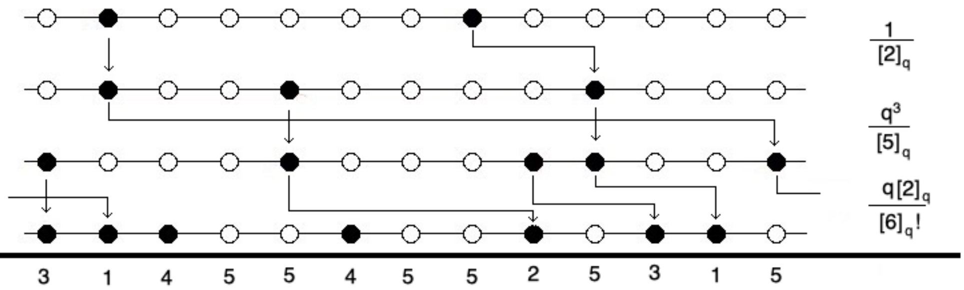


Figure 4.5: An LMLQ of type $(2, 1, 2, 2, 6)$ with weight $\frac{q^4}{[6]_q![5]_q}$

Finally, we use the projection principle to prove the main result of this chapter. Let $m_n = (1, \ldots, 1)$ and let Ω_n be the set of all multiline queues of type m_n. To compute the correlation $c^q_{ij}(n)$ from Theorem 4.2 for $i, j \in [n]$, we look at all the LMLQ of type m_n that project to a word with i in the first place and j in the second place. The projection principle states that the particles of a type larger than i look the same to a particle of type i. A similar argument holds for all particles of type smaller than i. Therefore, similar to Section 3.2, we can lump the multispecies PASEP of type m_n to the multispecies PASEP of type $m_{s,t} = (s, t, n - s - t)$. Given two particles with labels i and j, the particle with the lesser of the two labels becomes a 2 and that with the greater label becomes a 3 in the PASEP of type $m_{s,t}$. Let $\Omega_{s,t}(n)$ be the set of multiline queues of type $m_{s,t}$ and let

$$\begin{aligned} T_<(s,t) =& \mathbb{P}\{\omega_1 = 2,\, \omega_2 = 3\}, \quad \text{and} \\ T_>(s,t) =& \mathbb{P}\{\omega_1 = 3,\, \omega_2 = 2\}, \end{aligned}$$

where ω is a random word in the state space of the multispecies PASEP of type $m_{s,t}$. Let $i < j$. Then, by the projection principle we have

$$T_<(s,t) = \sum_{j=s+t+1}^{n} \sum_{i=s+1}^{s+t} c^q_{ij}(n),$$

and using the principle of inclusion-exclusion we get

$$c^q_{i,j}(n) = T_<(i-1, j-i) - T_<(i, j-i-1) - T_<(i-1, j-i+1) + T_<(i, j-i). \quad (4.2.3)$$

Similarly, for $i > j$ we have

$$T_{>}(s,t) = \sum_{i=s+t+1}^{n} \sum_{j=s+1}^{s+t} c_{ij}^{q}(n),$$

and hence we get

$$c_{i,j}^{q}(n) = T_{>}(j-1, i-j) - T_{>}(j, i-j-1) - T_{>}(j-1, i-j+1) + T_{>}(j, i-j). \quad (4.2.4)$$

4.3 Proof of Theorem 4.2

We first prove Theorem 4.2 for the case $i > j$. Let M be a multiline queue of type $m_{s,t}$ and (r,p) be the coordinate of p^{th} site in the r^{th} row such that $M_{(r,p)} \in \{\circ, \bullet\}$ denotes the occupancy status of the site at (r,p). We use (4.2.4) to solve for $c_{i,j}$. To compute $T_{>}(s,t)$, we only need to consider the multiline queues in $\Omega_{s,t}(n)$ which have either of the following structures:

$$\begin{array}{cccccc} \circ & \circ & . & . & \cdots & . \\ \circ & \bullet & . & . & \cdots & . \\ \hline 3 & 2 & . & . & \cdots & . \end{array} \qquad \text{or} \qquad \begin{array}{cccccc} \bullet & \circ & . & . & \cdots & . \\ \circ & \bullet & . & . & \cdots & . \\ \hline 3 & 2 & . & . & \cdots & . \end{array}$$

This holds because an unoccupied site is labelled 3, hence $M_{2,1} = \circ$ to ensure $\omega_1 = 3$. Also, $M_{2,2} = \bullet$ so that $\omega_2 \neq 3$. Further, if $M_{1,2} = \bullet$, then we have a trivial link at the second site in M giving $\omega_2 = 1$. Therefore, $M_{1,2} = \circ$.

Before computing the weights contributed by the LMLQs in the above two cases, we first consider the set $\Theta_{s,t}(k)$ of multiline queues M of type $(s, t, k-s-t)$ such that there is no $\bullet$ at the same site in both the rows in M. In other words, there are no trivial links in the LMLQs in $\Theta_{s,t}(k)$. Let $\eta_{s,t}(k)$ be the total weight of all the LMLQs in $\Theta_{s,t}(k)$ that project to a word beginning with 2. By the same argument as in the previous paragraph, this requires that we only consider the multiline queues in $\Theta_{s,t}(k)$ that begin with $\binom{\circ}{\bullet}$. Let $C_{s,t}(k)$ be the number of such multiline queues, i.e., $C_{s,t}(k) = |\{M' \in \Theta_{s,t}(k) : M' \text{ begins with } \binom{\circ}{\bullet}\}|$. We have

$$C_{s,t}(k) = \binom{k-1}{s,\ s+t-1,\ k-2s-t},$$

because the first column is fixed and we only have to select s $\binom{\bullet}{\circ}$ and $s+t-1$ $\binom{\circ}{\bullet}$

columns from $k-1$ columns. Next, we state a theorem computing $\eta_{s,t}(k)$.

Theorem 4.5. *For $s, t \geq 1$ and $s+t \leq k$, we have*

$$\eta_{s,t}(k) = \frac{t}{s+t}\binom{k-1}{s,\ s+t-1,\ k-2s-t}. \tag{4.3.1}$$

Remark 4.6. It is interesting to note that despite being a sum of link weights which are q-fractions, $\eta_{s,t}(k)$ adds up to a rational number. There is no dependence on q and it is not trivial to see why!

Before looking at the proof of Theorem 4.5, let us first consider an example for $\Theta_{1,2}(4)$.

Example 4.7. We have $C_{1,2}(4) = 3$. We consider below these three multiline queues from the set $\Theta_{1,2}(4)$ that begin with $\left(\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}\right)$ and list out all the possible projected words along with their weights. According to Theorem 4.5,

$$\eta_{1,2}(4) = \frac{2}{3}\binom{3}{1}\binom{2}{2} = 2.$$

$$\begin{array}{ccccc}
\circ & \bullet & \circ & \circ & \\
\bullet & \circ & \bullet & \bullet & wt \\
\hline
\textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{\frac{1}{[3]_q}} \\
\textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{2} & \textcolor{red}{1} & \textcolor{red}{\frac{q}{[3]_q}} \\
1 & 3 & 2 & 2 & \frac{q^2}{[3]_q}
\end{array}
\qquad
\begin{array}{ccccc}
\circ & \circ & \bullet & \circ & \\
\bullet & \bullet & \circ & \bullet & wt \\
\hline
\textcolor{red}{2} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{1} & \textcolor{red}{\frac{1}{[3]_q}} \\
1 & 2 & 3 & 2 & \frac{q}{[3]_q} \\
\textcolor{red}{2} & \textcolor{red}{1} & \textcolor{red}{3} & \textcolor{red}{2} & \textcolor{red}{\frac{q^2}{[3]_q}}
\end{array}
\qquad
\begin{array}{ccccc}
\circ & \circ & \circ & \bullet & \\
\bullet & \bullet & \bullet & \circ & wt \\
\hline
1 & 2 & 2 & 3 & \frac{1}{[3]_q} \\
\textcolor{red}{2} & \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} & \textcolor{red}{\frac{q}{[3]_q}} \\
\textcolor{red}{2} & \textcolor{red}{2} & \textcolor{red}{1} & \textcolor{red}{3} & \textcolor{red}{\frac{q^2}{[3]_q}}
\end{array}$$

Note that the words in red are the ones that begin with a 2 and the sum of the weights of LMLQs in $\Theta_{1,2}(4)$ that project to such a word is $\eta_{1,2}(4) = \frac{2(1+q+q^2)}{[3]_q} = 2$.

In addition, we make the following observations about this example. Adding the weights from all the LMLQs of a multiline queue gives 1. For instance, the sum of all the weights in each row is 1 for all three multiline queues. Next, rotating a linked multiline queue rotates the projected word by the same distance while preserving the weight. In each row above, the configurations in each column are rotations of one another and have the same weight. Based on these observations, we prove these properties for a more general class of multiline queues.

Lemma 4.8. *Let M be a multiline queue in $\Theta_{s,t}(k)$. The following holds true for M.*

(1) The sum of weights of all the linked multiline queues of M is 1.

(2) Rotating M while keeping the same links rotates the projected word by the same distance, and the weights of the corresponding linked multiline queues remain unchanged.

Proof. We prove (1) by induction on s. Let $s = 1$. There are $t+1$ possible links from the only $\bullet$ in the first row to a particle in the second row. This accounts for $t+1$ LMLQs of M; each corresponding to one of these possibilities and they have weights $q^{i-1}/[t+1]_q$ for $i \in \{1, \ldots, t+1\}$. Adding these weights for all i, we get 1.

Let us assume (1) is true for $s-1$. Let M have s $\bullet$'s in the first row and $(s+t)$ $\bullet$'s in the second row at sites different from those with $\bullet$'s in the first row. Let the occupied sites in the second row be labelled as $b_1, \ldots, b_{s+t}$. We can link the particles of the first row to the particles in the second row in any order, in particular from left to right. Let the leftmost $\bullet$ (say at site a) in the first row be linked to the particle at site b_i, where $1 \leq i \leq s+t$. The weight of this link is $q^{i-1}/[s+t]_q$. Constructing links for the remaining $\bullet$'s in the first row is the same as constructing links in a multiline queue M_d which is obtained from the multiline queue M by deleting the columns a and b_i. Note that $M_d \in \Theta_{s-1,t}(k-2)$ and the sum of weights of all the LMLQs of M_d is 1. That is, the sum of all the LMLQs of M where $a \to b_i$ is $1 \cdot (q^{i-1}/[s+t]_q)$. Summing over all $i \in [s+t]$, we get (1).

To prove (2), recall that the projected word describes the label of each site in the second row. Hence, rotating both rows of the MLQ simultaneously while keeping the links preserved only rotates the projected word without changing the weights of any links. $\square$

Proof of Theorem 4.5. Let $\mathcal{S}$ be the set of all the linked multiline queues in $\Theta_{s,t}(k)$ which begin with $\binom{\circ}{\bullet}$ and project to a word beginning with 2. Recall that $\eta_{s,t}(k)$ is the sum of the weights of all LMLQs in $\mathcal{S}$. Let $P \in \mathcal{S}$ has the following structure

$$P = \begin{array}{cccccc} \circ & . & . & \ldots & . & \\ \bullet & . & . & \ldots & . & . \\ \hline 2 & . & . & \ldots & . & \end{array}$$

Further, recall that $C_{s,t}(k)$ is the cardinality of set $\mathcal{M} = \{M \in \Theta_{s,t}(k) : M \text{ begins with } \binom{\circ}{\bullet}\}$. Consider any $M \in \mathcal{M}$. The weights from all the LMLQs of M sum up to 1 by Lemma 4.8 (2). Note that not all of these LMLQs belong to $\mathcal{S}$ as not all of them project to a word with a 2 in the first position. However, each LMLQ of M has exactly t rotations that are in $\mathcal{S}$ because there are t $\bullet$s in the second row that are not linked. Let Γ be the collection of all the rotations of all the linked multiline queues in $\mathcal{M}$ that belong to $\mathcal{S}$. Note that Γ is a multiset. By Lemma 4.8 (1 and 2), the sum of the weights of all the linked multiline queues in Γ is $tC_{s,t}$,

Further, note that for each LMLQ in $\mathcal{S}$, there are $(s+t)$ rotations which start with

$\left(\begin{smallmatrix}\circ\\ \bullet\end{smallmatrix}\right)$. In other words, each configuration of $\mathcal{S}$ is obtained as a rotation of $(s+t)$ different LMLQs in $\mathcal{M}$. Therefore, each linked multiline queue of $\mathcal{S}$ has $(s+t)$ copies in Γ. Hence, the sum of the weights of all the linked multiline queues in Γ is $(s+t)\eta_{s,t}(k)$. We use the method of counting in two ways to get the equation

$$(s+t)\eta_{s,t}(k) = tC_{s,t}(k), \tag{4.3.2}$$

thereby completing the proof. $\square$

To compute $T^{>}_{s,t}(n)$, recall that we only need to consider the LMLQs with either of the following structure:

$$\text{(A)} \quad \begin{array}{cccccc} \circ & \circ & . & . & \dots & . \\ \circ & \bullet & . & . & \dots & . \\ \hline 3 & 2 & . & . & \dots & . \end{array} \qquad \text{or} \qquad \text{(B)} \quad \begin{array}{cccccc} \bullet & \circ & . & . & \dots & . \\ \circ & \bullet & . & . & \dots & . \\ \hline 3 & 2 & . & . & \dots & . \end{array}$$

Let $W^A_{s,t}(n)$ and $W^B_{s,t}(n)$ denote the sum of the weights of the linked multiline queues of type (A) and (B) respectively in $\Omega_{s,t}(n)$ for which the corresponding word begins with (3,2). For $X \in \{A, B\}$, let $U^X_{s,t}(n) \subset \Omega_{s,t}(n)$ be the set of LMLQs of type (X) that have no $\bullet$ at the same position in both the rows. Now, let $\tau^X_{s,t}(n)$ be the weight contributed to $W^X_{s,t}(n)$ by the LMLQs from $U^X_{s,t}(n)$.

Lemma 4.9. *Let $s, t \geq 0$, $n > s+t$ and $X \in \{A, B\}$. $W^X_{s,t}$ and $\tau^X_{s,t}$ are related by the following equations.*

$$W^A_{s,t}(n) = \sum_{i=0}^{s} \binom{n-2}{i} \tau^A_{s-i,t}(n-i), \tag{4.3.3}$$

$$W^B_{s,t}(n) = \sum_{i=0}^{s-1} \binom{n-2}{i} \tau^B_{s-i,t}(n-i). \tag{4.3.4}$$

Proof. Links in any LMLQ can be constructed in any arbitrary order. So we construct all the trivial links first and then process the remaining particles from left to right. Since the weight of a trivial link is 1, the weight of an LMLQ is equal to the product of the weights of non-trivial links.

Let an arbitrary linked multiline queue contributing to $W^X_{s,t}(n)$ have i $\left(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\right)$ columns. Then, $i \in \{0, \dots, s\}$ for $X = A$ and $i \in \{0, \dots, s-1\}$ for $X = B$. There are $\binom{n-2}{i}$ choices for these i columns. Deleting these columns results in an LMLQ of type $(s-i, t, n-i)$ that belongs to $U^X_{s-i,t}(n-i)$. Summing over all possible values of i, we get Lemma 4.9. $\square$

Note that the linked multiline queues contributing to $\tau^A_{s,t}(n)$ are in bijection to those of $\eta_{s,t}(n-1)$ because the first column $\binom{\circ}{\circ}$ of a multiline queue of type (A) does not contribute to any link or to the weight of the configuration. So, $\tau^A_{s,t}(k) = \eta_{s,t}(k-1)$ for all $k > 1$. Substituting this equation in (4.3.3):

$$\begin{aligned} W^A_{s,t}(n) &= \sum_{i=0}^{s} \binom{n-2}{i} \eta_{s-i,t}(n-i-1) \\ &= \frac{t(n-2)!(n-1)!}{s!(n-s-1)!(s+t)!(n-s-t-1)!}. \end{aligned}$$

Therefore,

$$\frac{W^A_{s,t}(n)}{\binom{n}{s}\binom{n}{s+t}} = \frac{t(n-s)(n-s-t)}{n^2(n-1)}.$$

To find $\tau^B_{s,t}$, we start creating links from the second particle in the first row and move rightwards along the ring. We construct the link of $\bullet$ at the first site at the end and it has $t+1$ particles available in the second row to bully. Hence, the weight of the link of this particle is $q^{h-1}/[t+1]_q$ for some $h \in \{2, \dots, t+1\}$ because for $h = 1$, we get $\omega_2 = 1$. Removing the first column from any linked multiline queue in $U^B_{s,t}(n)$ gives a linked multiline queue of type $(s-1, t+1, n-1)$ that projects to a word beginning with 2 and has no $\bullet$ at the same position in both the rows. Recall that the sum of weights of these LMLQs is equal to $\eta_{s-1,t+1}(n-1)$. Therefore,

$$\begin{aligned} \tau^B_{s,t}(n) &= \eta_{s-1,t+1}(n-1) \sum_{h=2}^{t+1} \frac{q^{h-1}}{[t+1]_q} \\ &= \frac{t+1}{s+t}\binom{n-2}{s-1}\binom{n-s-1}{s+t-1}\left(1 - \frac{1}{[t+1]_q}\right). \end{aligned}$$

Substituting this in (4.3.4),

$$\begin{aligned} W^B_{s,t}(n) &= \sum_{i=0}^{s-1} \binom{n-2}{i} \tau^B_{s-i,t}(n-i) \\ &= \frac{(t+1)(n-2)!(n-1)!}{(s-1)!(n-s)!(s+t)!(n-s-t-1)!}\left(1 - \frac{1}{[t+1]_q}\right). \end{aligned}$$

Therefore,

$$\frac{W^B_{s,t}(n)}{\binom{n}{s}\binom{n}{s+t}} = \frac{s(t+1)(n-s-t)}{n^2(n-1)}\left(1 - \frac{1}{[t+1]_q}\right).$$

We have $\binom{n}{s}\binom{n}{s+t} T^{>}_{s,t}(n) = W^A_{s,t}(n) + W^B_{s,t}(n)$. If $t = 0$ or $s + t = n$, the formula is trivially satisfied for either of these cases as it is impossible for the projected word to start with $(3, 2)$. Let $s = 0$. Then, there is no type (B) multiline queues and the second row of type (A) multiline queues look like $\circ \bullet \ldots .$, hence there are $\binom{n-2}{t-1}$ multiline queue's and each contribute weight 1. So, $T^{>}_{0,t}(n) = \frac{\binom{n-2}{t-1}}{\binom{n}{t}} = \frac{t(n-t)}{n(n-1)}$. Thus, we can write

$$
\begin{aligned}
T^{>}_{s,t}(n) &= \frac{t(n-s)(n-s-t)}{n^2(n-1)} + \frac{s(t+1)(n-s-t)}{n^2(n-1)}\left(1 - \frac{1}{[t+1]_q}\right) \\
&= \frac{t(n-s)(n-s-t)}{n^2(n-1)} + \frac{s(t+1)(n-s-t)q[t]_q}{n^2(n-1)[t+1]_q},
\end{aligned} \tag{4.3.5}
$$

for all $0 \leq s, t \leq n, s+t \leq n$.

Remark 4.10. Interestingly, the weight contributed by the type (A) multiline queues does not depend on the value of q. Further, the formula in (4.3.5) is consistent with $T^{>}_{s,t}(n)$ in [20] for the case $q = 0$, i.e., the multispecies TASEP on a ring.

The proof is completed by substituting (4.3.5) in (4.2.4). We get

$$
\begin{aligned}
c^q_{i,j}(n) = \frac{2(i-j)}{n^2(n-1)} &+ \frac{(i-j+1)(2j(n-i)+i+j-n-1)q[i-j]_q}{n^2(n-1)[i-j+1]_q} \\
- \frac{(i-j+2)(j-1)(n-i)q[i-j+1]_q}{n^2(n-1)[i-j+2]_q} &- \frac{j(i-j)(n-i+1)q[i-j-1]_q}{n^2(n-1)[i-j]_q},
\end{aligned} \tag{4.3.6}
$$

when $i > j+1$. For $i = j+1$, we add terms corresponding to $T >_{j,i-j-1}$ from (4.3.5) to (4.3.6), which are

$$
\frac{j(n-j)}{n^2(n-1)} - \frac{j(i-j)(n-i+1)}{n^2(n-1)[i-j]_q}.
$$

This proves Theorem 4.2 for the case $1 \leq j < i \leq n$.

This case $i < j$ is now solved analogously. Recall $T^{<}_{s,t}$ denotes the probability $\mathbb{P}\{\omega_1 = 2,\ \omega_2 = 3\}$ where ω is the word projected by a random multiline queue in $\Omega_{s,t}$. The multiline queues that project to a word beginning with $(2,3)$ are of either of the following types.

$$
(C)\ \begin{array}{cccccc} \circ & \circ & . & . & \ldots & . \\ \bullet & \circ & . & . & \ldots & . \\ \hline 2 & 3 & . & . & \ldots & . \end{array} \qquad \text{or} \qquad (D)\ \begin{array}{cccccc} \circ & \bullet & . & . & \ldots & . \\ \bullet & \circ & . & . & \ldots & . \\ \hline 2 & 3 & . & . & \ldots & . \end{array}
$$

We compute the weights of the two cases separately as follows:

$$
(C)\ \begin{array}{cccccc} \circ & \circ & . & . & \ldots & . \\ \bullet & \circ & . & . & \ldots & . \\ \hline 2 & 3 & . & . & \ldots & . \end{array}
$$

Let $W^C_{s,t}(n)$ denote the sum of the weights of the LMLQs of MLQs of type (C)

in $\Omega_{s,t}(n)$ for which the corresponding word begins with $(2,3)$. Note that interchanging the first two columns of an LMLQ of type (C) does not change its weight. Hence, $W^C_{s,t}(n) = W^A_{s,t}(n)$.

$$\text{(D)}\quad \begin{array}{cccccc} \circ & \bullet & . & . & \ldots & . \\ \bullet & \circ & . & . & \ldots & .\,. \\ \hline 2 & 3 & . & . & \ldots & . \end{array}$$

Let $W^D_{s,t}(n)$ denote the sum of the weights of the linked multiline queues of MLQs of type (D). Let $U^D_{s,t}(n)$ be the set of those LMLQs which have no $\binom{\bullet}{\bullet}$ columns. Let $\tau^D_{s,t}(n)$ be the weight contributed to $W^D_{s,t}$ by the linked multiline queues in $U^D_{s,t}(n)$. We have a similar relation between $W^D_{s,t}(n)$ and $\tau^D_{s,t}(n)$ as we had for $W^B_{s,t}(n)$ and $\tau^B_{s,t}(n)$ and it can be proved using the same arguments as in Lemma 4.9. We have

$$W^D_{s,t} = \sum_{i=0}^{s-1} \binom{n-2}{i} \tau^D_{s-i,t}(n-i). \tag{4.3.7}$$

To find $\tau^D_{s,t}$, we start creating links from the second $\bullet$ in the first row and move rightwards along the ring. We construct the link of leftmost $\bullet$ in the first row in the end and it has a weight equal to $q^{h-1}/[t+1]_q$ for some $h \in \{1, \ldots, t\}$ because for $h = t+1$, we get $\omega_1 = 1$. Deleting the second column from any LMLQ in $U^D_{s,t}(n)$ gives an LMLQ of type $(s-1, t+1, n-1)$ that begins with $\substack{\circ \\ \bullet}$, has no $\bullet$ at the same position in both the rows and projects to a word that starts with 2. The sum of weights of the linked multiline queues is again equal to $\eta_{s-1,t+1}(n-1)$. Therefore,

$$\begin{aligned} \tau^D_{s,t}(n) &= \eta_{s-1,t+1}(n-1) \sum_{h=1}^{t} \frac{q^{h-1}}{[t+1]_q} \\ &= \frac{t+1}{s+t} \binom{n-2}{s-1} \binom{n-s-1}{s+t-1} \left(1 - \frac{q^t}{[t+1]_q}\right). \end{aligned}$$

Substituting this in (4.3.7), we get

$$\begin{aligned} W^D_{s,t}(n) &= \sum_{i=0}^{s-1} \binom{n-2}{i} \tau^D_{s-i,t}(n-i) \\ &= \frac{(t+1)(n-2)!(n-1)!}{(s-1)!(n-s)!(s+t)!(n-s-t-1)!} \left(1 - \frac{q^t}{[t+1]_q}\right). \end{aligned}$$

Therefore,

$$\frac{W^D_{s,t}(n)}{\binom{n}{s}\binom{n}{s+t}} = \frac{s(t+1)(n-s-t)}{n^2(n-1)} \left(1 - \frac{q^t}{[t+1]_q}\right).$$

Once again we have, $\binom{n}{s}\binom{n}{s+t}T^{<}_{s,t} = W^{C}_{s,t} + W^{D}_{s,t}$, i.e.,

$$T^{<}_{s,t}(n) = \frac{(s+tn)(n-s-t)}{n^2(n-1)} - \frac{s(t+1)(n-s-t)q^t}{n^2(n-1)[t+1]_q}, \tag{4.3.8}$$

and substituting this in (4.2.3),

$$\begin{aligned} c^q_{i,j}(n) = \frac{1}{n^2} &- \frac{(j-i+1)(2i(n-j)+i+j-n-1)q^{(i-j)}}{n^2(n-1)[j-i+1]_q} \\ &+ \frac{i(j-i)(n-j+1)q^{(j-i-1)}}{n^2(n-1)[j-i]_q} + \frac{(i-1)(j-i+2)(n-j)q^{(j-i+1)}}{n^2(n-1)[j-i+2]_q}, \end{aligned} \tag{4.3.9}$$

when $i > j+1$. For $i = j+1$, we add terms corresponding to $T^{>}_{j,i-j-1}$ from (4.3.8) to (4.3.9), which are

$$\frac{j(n-j)}{n^2(n-1)} - \frac{j(i-j)(n-i+1)}{n^2(n-1)[i-j]^q}.$$

This proves Theorem 4.2 for the case $1 \leq j < i \leq n$.

Chapter 5

Correlations in the multispecies B-TASEP

Consider any finite Weyl group W. Thomas Lam [71] studied random reduced words in affine Weyl group $\widetilde{W}$ corresponding to W. This is equivalent to a random walk on the alcoves of $\widetilde{W}$ under the condition that the walk enters a new alcove at each step and never crosses any hyperplane in the Coxeter arrangement twice. Lam defined a finite state Markov chain on W whose stationary distribution, say π_W, can be formulated in terms of the limiting direction of the random walk. In particular, for the affine Weyl group of type A, he conjectured a closed formula for the limiting direction in terms of a certain Markov chain on the set of permutations. Ayyer and Linusson [20] proved Lam's conjecture for the Weyl group of type A by investigating correlations in another Markov chain, the multispecies TASEP on a ring (see [19]). Another study that was conducted to find the limiting direction when W in a classical Weyl group was done by Aas, Ayyer, Linusson and Potka [2] where they focused on studying the corresponding asymmetric exclusion process for the affine Weyl groups of type B, C and D and computed the limiting directions of the random reduced walks for respective groups.

The key to finding these limiting directions is to compute certain correlations in the corresponding TASEPs. To that end, the authors of the paper [2] conjectured an interesting formula for the two-point correlations of adjacent particles on the last two sites of the multispecies B-TASEP [2, Conjecture 3.5]. In this chapter, we study these correlations aiming to solve the aforementioned conjecture.

5.1 Background

We first look at the exclusion process known as the multispecies B-TASEP (see [1]). Let $[\pm n]$ denote the set $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$. This process is defined on a lattice with n sites such that there is at most one particle at each site. Each particle is labelled with numbers from the set $[\pm n]$ and there is the natural order among the particles defined by these labels. The state space for this process consists of words on length n where exactly one of i or $-i$ is present, for all i. A particle with a higher label exchanges positions with a particle of a smaller label only if the former of the two is towards the left of the latter. We also write $-i$ as $\bar{i}$. The dynamics of this process are as follows: An edge joining two sites i and $i+1$ is chosen with probability $\frac{1}{n}$ for $i \in \{0, \dots, n-2\}$, and with probability $\frac{1}{2n}$ for $i \in \{n-1, n\}$. When the chosen edge lies in the bulk of the lattice, i.e., for $1 \le i \le n-1$, the transition exchanges the particles at adjoining sites. For $i = 0$, the transition only changes the sign of the first particle. If finally $i = n$, the last two particles are exchanged and their signs are reversed. The rules are given in Table 5.1.

First Site		Bulk		Last two Sites	
Probability	Transition	Probability	Transition	Probability	Transition
$\frac{1}{n}$	$\bar{i} \to i$	$\frac{1}{n}$	$mk \to km$	$\frac{1}{2n}$	$ji \to \overline{ij}$ $ji \to ij$ $j\bar{i} \to i\bar{j}$ $j\bar{i} \to \bar{i}j$ $ij \to \overline{ji}$ $ij \to ji$ $\bar{i}j \to \bar{j}i$ $\overline{ij} \to \overline{ji}$

Table 5.1: Transition rules for the multispecies B-TASEP where $k < m \in [\pm n]$ and $i < j \in [n]$.

Let Π_B denote the stationary distribution of this exclusion process. Let $\langle i, j \rangle$ denote the probability in Π_B that the last two sites in the multispecies B-TASEP are occupied by particles labelled i and j respectively. We know from [2, Theorem 7.11] that $\langle i, \bar{j} \rangle = \langle i, j \rangle$. The following conjecture states the two-point correlations for the last two positions.

Conjecture 5.1. *[2, Conjecture 3.5] For $i, j \in [\pm n]$, we have*

1. *For $3 \le i \le n$, $1 \le j \le i-2$,*

$$\langle \bar{i}, \bar{j} \rangle = \frac{1}{(2n)^2}.$$

2. *For* $1 \le j \le n-1$,
$$\langle \overline{j+1}, \bar{j} \rangle = \frac{1}{(2n)^2} + \frac{n^2 - j^2}{4n^2(2n-1)}.$$

3. *For* $1 \le i \le n-1$, $i+1 \le j \le n$,
$$\langle \bar{i}, \bar{j} \rangle = \frac{j-i}{2n^2(2n-1)},$$
and for $1 \le i \le n-2$, $i+2 \le j \le n$,
$$\langle i, \bar{j} \rangle = \frac{i+j-1}{2n^2(2n-1)}.$$

4. *For* $1 \le j \le n-1$,
$$\langle j, \overline{j+1} \rangle = \frac{j(n^2 - j^2 + 2n - 2)}{2n^2(2n-1)(n-1)}.$$

5. *For* $2 \le i \le n$, $1 \le j \le i-1$,
$$\langle i, \bar{j} \rangle = \frac{3(i-j)(i+j-1)}{4n^2(2n-1)(n-1)}.$$

5.1.1 B-TASEP

B-TASEP is a two-species TASEP defined on a lattice with n sites with a fixed number of vacancies. The sites are labelled with the integers from the set $\{\bar{1}, 0, 1\}$. Let the number of vacant sites (labelled 0) be fixed to n_0. The dynamics for the process are similar to the multispecies case and the transition rules are given in Table 5.2.

First Site		Bulk		Last two Sites	
Probability	Transition	Probability	Transition	Probability	Transition
$\frac{1}{n}$	$\bar{1} \to 1$	$\frac{1}{n}$	$1\bar{1} \to \bar{1}1$ $10 \to 01$ $0\bar{1} \to \bar{1}0$	$\frac{1}{2n}$	$11 \to \bar{1}\bar{1}$ $1\bar{1} \to \bar{1}1$ $01 \to \bar{1}0$ $0\bar{1} \to \bar{1}0$ $10 \to 0\bar{1}$ $10 \to 01$

Table 5.2: Transitions for the B-TASEP

Let Ω_n and Ω_{n,n_0} be the respective state-spaces of the multispecies B-TASEP and the B-TASEP. We can lump the multispecies B-TASEP to the B-TASEP by defining a map $f : \Omega_n \to \Omega_{n,n_0}$ as follows:

$$f(\omega_1, \dots, \omega_n) = (f(\omega_1), \dots, f(\omega_n)),$$

where

$$f(i) = \begin{cases} 1, & \text{if } i \geq n_0, \\ \bar{1}, & \text{if } i \leq -n_0, \\ 0, & \text{if } -n_0 < i < n_0. \end{cases}$$

The partition function and the two-point correlations $\langle i, j \rangle_{n_0}$ for the last two sites in the B-TASEP are calculated in [2, Section 7.2] and are stated as follows.

Theorem 5.2. *[2, Theorem 7.10] For any $n \geq n_0 \geq 0$, the partition function for the B-TASEP is given by*

$$Z_{n,n_0} = \binom{2n}{n - n_0}.$$

Theorem 5.3. *[2, Theorem 7.11] Let $n \geq n_0 \geq 0$, the following table contains the values $Z_{n,n_0}.\langle i, j \rangle_{n_0}$ for $i, j \in \{\bar{1}, 0, 1\}$ in the B-TASEP.*

$i\backslash j$	$\bar{1}$	0	1
$\bar{1}$	$\binom{2n-2}{n-n_0-2}$	$C^{n+n_0-1}_{n-n_0-1}$	$\binom{2n-2}{n-n_0-2}$
0	$C^{n+n_0-2}_{n-n_0-1}$	$C^{n+n_0-3}_{n-n_0}$	$C^{n+n_0-2}_{n-n_0-1}$
1	$2\binom{2n-3}{n-n_0-2}$	$C^{n+n_0-2}_{n-n_0-1}$	$2\binom{2n-3}{n-n_0-2}$

Table 5.3: Values of $Z_{n,n_0}.\langle i, j \rangle_{n_0}$ for $i, j \in \{\bar{1}, 0, 1\}$

5.2 Results

In this section, we first define a three-species TASEP corresponding to the type B Weyl group. Let us call this process B_3-TASEP. Let $t = (k, \ell, m)$ be a tuple with non-negative integers and let $n = k + l + m$. A B_3-TASEP of type t is defined on a lattice with n sites with a fixed number of vacancies and two kinds of particles. Let Ω_t denote the state-space of this process. The sites are labelled with the integers from the set $\{\bar{2}, \bar{1}, 0, 1, 2\}$. Let the number of vacant sites (labelled 0) be m and the number of first-kind particles (labelled 1 or $\bar{1}$) and second-kind particles (labelled 2 or $\bar{2}$) be ℓ and k respectively. The dynamics of the process are similar to the multispecies case.

We can now lump the multispecies B-TASEP to the B_3-TASEP by defining a map $f_t : \Omega_n \to \Omega_t$ as follows:

$$f_t(\omega_1, \ldots, \omega_n) = (f_t(\omega_1), \ldots, f(\omega_n)),$$

where

$$f_t(i) = \begin{cases} 2, & \text{if } i \geq \ell + m, \\ 1, & \text{if } i \geq \ell, \\ \bar{1}, & \text{if } i \leq -\ell, \\ \bar{2}, & \text{if } i \leq -\ell - m, \\ 0, & \text{if } -\ell < i < \ell. \end{cases}$$

Let $\langle i, j \rangle_t$ the probability that particles labelled i and j are in the last two positions in a B_3-TASEP. Let us now define the corresponding row and column sums for the B_3-TASEP for $i \in \{-2, -1, 0, 1, 2\}$.

$$\mathrm{Col}_i(n) = \sum_{j=\bar{2}}^{2} \langle j, i \rangle_t, \qquad \mathrm{Row}_i(n) = \sum_{j=\bar{2}}^{2} \langle i, j \rangle_t.$$

Next, we define the corresponding up and down-hooks (see Figure 5.1) for the B_3-TASEP for $i \in \{0, 1, 2\}$.

$$\mathrm{DHook}_i(n) = \sum_{j=i+1}^{2} \langle i, \bar{j} \rangle_t + \langle j, \bar{i} \rangle_t, \qquad \mathrm{UHook}_i(n) = \sum_{j=i+1}^{2} \langle \bar{j}, i \rangle_t + \langle \bar{i}, j \rangle_t.$$

	$\bar{2}$	$\bar{1}$	0	1	2
$\bar{2}$					
$\bar{1}$					
0					
1					
2					

	$\bar{2}$	$\bar{1}$	0	1	2
$\bar{2}$					
$\bar{1}$					
0					
1					
2					

Figure 5.1: The down-hook $\mathrm{DHook}_1(n)$ in the first table and the up-hook $\mathrm{UHook}_0(n)$ in the second table is shaded.

We can also lump the B_3-TASEP to the B-TASEP in two different ways; one of them is identifying the two kinds of particles in the B_3-TASEP to a particle in the B-TASEP and identifying the vacant sites in both. This projects a B_3-TASEP of type $t = (k+\ell+m)$ to a B-TASEP on n sites with m vacancies. The other way is to identify both the first-kind particles and the vacancies of the B_3-TASEP with type t to the vacancies in the B-TASEP and the second-kind particles of the B_3-TASEP to the particles in the B-TASEP. The resulting process has $\ell+m$ vacancies. Using these lumpings and manipulating values from Table 5.3, we can now prove the following lemmas.

Lemma 5.4. *For a B_3-TASEP of type $t = (k, \ell, m)$ with $n = k + \ell + m$, we have*

$$\langle \bar{2}, \bar{2} \rangle_t = \frac{k(k-1)}{2n(2n-1)}, \tag{5.2.1}$$

$$\langle 0, 0 \rangle_t = \frac{(2m-2)(n+m)(n+m-1)}{2n(2n-1)(2n-2)}, \tag{5.2.2}$$

$$\langle 2, \bar{2} \rangle_t = \frac{k(k-1)(2n-k)}{2n(2n-1)(2n-2)}, \tag{5.2.3}$$

$$\langle 1, 0 \rangle_t + \langle 2, 0 \rangle_t = \frac{m(k+\ell)(n+m)(2n-k)}{n(2n-1)(2n-2)}. \tag{5.2.4}$$

Proof. We prove (5.2.1) using the lumping where we identify both the first-kind particles and the vacancies of the B_3-TASEP with type t to the vacancies in the B-TASEP and the second-kind particles of the B_3-TASEP to the particles in the B-TASEP of type $(k, \ell+m)$. Thus, we have

$$\begin{aligned} \langle \bar{2}, \bar{2} \rangle_t &= \langle \bar{1}, \bar{1} \rangle_{(\ell+m)} \\ &= \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}} = \frac{k(k-1)}{2n(2n-1)}. \end{aligned}$$

The remaining equations can be proved in a similar way. $\square$

Lemma 5.5. *For a B_3-TASEP of type $t = (k, \ell, m)$, we have*

$$\begin{aligned} Col_2(n) &= \frac{k}{2n}, & Col_1(n) &= \frac{\ell}{2n}, & Col_0(n) &= \frac{m}{n}, \\ Row_{\bar{2}}(n) &= \frac{k}{2n}, & Row_{\bar{1}}(n) &= \frac{\ell}{2n}, & Row_0(n) &= \frac{(k+\ell)m(k+\ell+2m)}{n(n-1)(2n-1)}, \\ UHook_0(n) &= \frac{m}{n}, & DHook_0(n) &= Row_0(n), & UHook_1(n) &= \frac{\ell(2k+\ell-1)}{2n(2n-1)}, \end{aligned}$$

$$Row_2(n) = \frac{k(k+2\ell+2m)(2k+\ell+m-2)}{n(n-1)(2n-1)},$$

$$Dhook_1(n) = \frac{\ell(k^2+(\ell-1)(\ell+2m)+k(k+3\ell+4m)}{n(2n-1)(2n-2)}. \tag{5.2.5}$$

Proof. We prove (5.2.5) by lumping the B_3-TASEP to the B-TASEP in both ways. We have

$$\begin{aligned}\mathrm{DHook}_1(n) &= \langle 1,\bar{2}\rangle_t + \langle 1,\bar{1}\rangle_t + \langle 2,\bar{1}\rangle_t,\\ &= \langle 1,\bar{1}\rangle_{(k+\ell)} - \langle 1,\bar{1}\rangle_k,\\ &= \frac{2\binom{2n-3}{m-2}}{\binom{2n}{m}} - \frac{2\binom{2n-3}{\ell+m-2}}{\binom{2n}{\ell+m}}.\end{aligned}$$

The remaining equations can be proved in a similar way. □

In addition to Lemma 5.4, we have conjectured the formulae for $\langle i,j\rangle_t$ for a few more of the values of $i \in \{\bar{2},\bar{1},0,1,2\}$ and $j \in \{\bar{2},\bar{1},0\}$ as given in Conjecture 5.6. This has been achieved by observing the values of these correlations for many different values of k,ℓ,m and n.

Conjecture 5.6. *We have*

$$\langle \bar{2},0\rangle_t = \frac{km}{2n^2}, \tag{5.2.6}$$

$$\langle 0,\bar{2}\rangle_t = \frac{km(2n-k)}{2n^2(2n-1)}, \tag{5.2.7}$$

$$\langle 0,\bar{2}\rangle_t + \langle 0,\bar{1}\rangle_t + \langle 1,\bar{1}\rangle_t + \langle 2,\bar{1}\rangle_t = \frac{(2n-k)(k+\ell)(\ell+2m)}{4n^2(2n-1)}. \tag{5.2.8}$$

Assuming Conjecture 5.6, and using Lemma 5.4 and Lemma 5.5, we can find expressions for many other $\langle i,j\rangle_t$ as stated in the result below.

Lemma 5.7. *Consider a B_3-TASEP of type $t=(k,\ell,m)$. Assuming Conjecture 5.6 we have*

$$\langle \bar{2},\bar{1}\rangle_t = \frac{k(m+n+2\ell)}{4n^2(2n-1)}, \tag{5.2.9}$$

$$\langle \bar{1},0\rangle_t = \frac{km+n(k+\ell+2\ell m)}{2n^2(2n-1)}, \tag{5.2.10}$$

$$\langle 0,\bar{1}\rangle_t = \frac{2m(k(2n-k)+\ell n(\ell+2m)}{2n^2(2n-1)(2n-2)}, \tag{5.2.11}$$

$$\langle \bar{1},\bar{2}\rangle_t + \langle \bar{1},\bar{1}\rangle_t = \frac{2\ell n(k+\ell-1)-k(m+n)}{4n^2(2n-1)}, \tag{5.2.12}$$

$$\langle 1,\bar{1}\rangle_t + \langle 2,\bar{1}\rangle_t = \frac{(k+\ell)(k(\ell+2m-n(\ell+4m))+2n(n-m-1)(\ell+2m))}{4n^2(2n-1)(n-1)}. \tag{5.2.13}$$

The findings from this section are summarised in Table 5.4. The expressions written in black are derived from using earlier results. The expressions in red are the ones that we are able to give the conjecture for. Finally, the formulae written in blue are the consequences of the Conjecture 5.6.

$i\backslash j$	$\bar{2}$	$\bar{1}$	0
$\bar{2}$	$\frac{k(k-1)}{2n(2n-1)}$	$\frac{k(m+n+2\ell n)}{4n^2(2n-1)}$	$\frac{km}{2n^2}$
$\bar{1}$	(5.2.12)	(5.2.12)	$\frac{km+n(k+\ell+2lm)}{2n^2(2n-1)}$
0	$\frac{km(2n-k)}{2n^2(2n-1)}$	$\frac{2m(k(2n-k)+\ell n(\ell+2m))}{2n^2(2n-1)(2n-2)}$	$\frac{(2m-2)(n+m)(n+m-1)}{2n(2n-1)(2n-2)}$
1	(5.2.5)-(5.2.13)	(5.2.13)	(5.2.4)
2	$\frac{k(k-1)(2n-k)}{2n(2n-1)(2n-2)}$	(5.2.13)	(5.2.4)

Table 5.4: Values for $\langle i, j\rangle_t$ for $i, j \in \{\overline{2}, \overline{1}, 0, 1, 2\}$

These formulas bring us one step closer to proving Conjecture 5.1 using the lumping f_t as the two-point correlations $\langle i, j\rangle$ in the multispecies B-TASEP can be written exactly in terms of $\langle i, j\rangle_t$ in the B_3-TASEP using the inclusion-exclusion principle.

Chapter 6

Correlations in general multispecies TASEP

6.1 Introduction

Ayyer and Linusson studied the correlations of two or more particles in a multispecies TASEP on a ring with finite sites. In particular, they studied the correlations of the first two sites on the ring which has exactly one particle of each type. In this chapter, we generalise their result to a multispecies TASEP such that each type has an arbitrary number of particles.

Let $m = (m_1, \dots, m_n)$. Let $M = (M_1, \dots, M_n)$ be a tuple such that $M_i = m_1 + \cdots + m_i$, for $1 \le i \le n$ and let $N = M_n$. The states of the multispecies TASEP of type m are words of length N with each letter i occurring m_i times. Let $\hat{E}_{ij} = \mathbb{P}\{\omega_1 = i, \omega_2 = j\}$ denote the probability that the particles labelled i and j are in the first two places in the steady state distribution of the TASEP with type m. We prove the following main result.

Theorem 6.1. *For $1 \le i \le j \le n$, we have*

$$\hat{E}_{i,j}(n) = \begin{cases} \frac{m_i m_j}{N^2}, & i < j-1 \\ \frac{m_i m_{i+1}}{N^2} + \frac{M_i(N-M_i)}{N^2(N-1)}, & i = j-1 \\ \frac{M_i(m_i-1)(N-M_{i-1})}{N^2(N-1)}, & i = j \end{cases},$$

and for $1 \le j < i \le n$,

$$\hat{E}_{i,j}(n) = \frac{m_j\left\{\binom{M_i+1}{2} - \binom{M_{i-1}+1}{2}\right\} - m_i\left\{\binom{M_j+1}{2} - \binom{M_{j-1}+1}{2}\right\}}{N\binom{N}{2}}.$$

Once again, we use the projection principle and theory of lumping to prove Theorem 6.1. Consider $\langle 1^{\mathrm{n}} \rangle = (1, \ldots, 1)$. A multispecies TASEP of type $\langle 1^{\mathrm{n}} \rangle$ on N sites can be projected to any multispecies TASEP of type m such that $N = \sum m_i$. We will see this in detail in Section 6.2.

6.2 Proof of Theorem 6.1

Let $m_i = 1$ for $i \in [N]$. Let us consider a multispecies TASEP of type $\langle 1^N \rangle$. Denote by $E_{i,j}(N)$ the probability that i and j are in the first and the second places of the ring respectively in the steady state distribution of the multispecies TASEP with type $\langle 1^N \rangle$. Then we have,

Theorem 6.2. *[20, Theorem 4.2] We have for $i, j \in [N]$,*

$$E_{i,j}(N) = \begin{cases} \frac{i-j}{N\binom{N}{2}}, & \text{if } i > j, \\ \frac{1}{N^2} + \frac{i(N-i)}{N^2(N-1)}, & \text{if } i = j-1, \\ \frac{1}{N^2}, & \text{if } i < j-1. \end{cases}$$

Theorem 6.3. *Consider an n-tuple $m = (m_1, \ldots, m_n)$. Let $M = (M_i : 1 \leq i \leq n)$ where $M_i = m_1 + \cdots + m_i$ and let $M_0 = 0, M_n = N$. Then for $i, j \in [n]$,*

$$\hat{E}_{i,j}(n) = \sum_{d=M_{j-1}+1}^{M_j} \sum_{c=M_{i-1}+1}^{M_i} E_{c,d}(N). \tag{6.2.1}$$

Proof. This result is proved using the colouring argument. Consider a multispecies TASEP Θ of type $\langle 1^{\mathrm{n}} \rangle$ that is projected to a multispecies TASEP Ξ of type m by identifying integers $1, \ldots, m_1$ in Θ to 1 in Ξ, $m_1 + 1, \ldots, m_1 + m_2$ in Θ to 2 in Ξ and so on till $M_{n-1} + 1, \ldots, M_n$ in Θ to n in Ξ. The result follows by adding the values of $E_{c,d}$ over appropriate ranges of $c, d \in [N]$. $\square$

Proof of Theorem 6.1. We can now use (6.2.1) to prove the main result of the chapter.

(a) The case $i < j - 1$ is straightforward as $1/N^2$ is summed $(M_i - M_{i-1})(M_j - M_{j-1}) = m_i m_j$ times.

(b) $i = j - 1$

$$\begin{aligned}
\hat{E}_{i,i+1}(n) &= \sum_{d=M_i+1}^{M_{i+1}} \sum_{c=M_{i-1}+1}^{M_i} E_{c,d}(N) \\
&= \sum_{d=M_i+2}^{M_{i+1}} \sum_{c=M_{i-1}+1}^{M_i} \frac{1}{N^2} + \sum_{c=M_{i-1}+1}^{M_i-1} \frac{1}{N^2} + E_{M_iM_i+1}(N) \\
&= \frac{m_i(m_{i+1}-1)+(m_i-1)}{N^2} + \frac{1}{N^2} + \frac{M_i(N-M_i)}{N^2(N-1)} \\
&= \frac{m_i m_{i+1}}{N^2} + \frac{M_i(N-M_i)}{N^2(N-1)}.
\end{aligned}$$

(c) $i > j$

$$\begin{aligned}
\hat{E}_{i,j}(n) &= \sum_{d=M_j+1}^{M_{j+1}} \sum_{c=M_i+1}^{M_{i+1}} E_{c,d}(N) \\
&= \frac{1}{N\binom{N}{2}} \sum_{d=M_j+1}^{M_{j+1}} \sum_{c=M_i+1}^{M_{i+1}} (c-d) \\
&= \frac{1}{N\binom{N}{2}} \left\{ \sum_{d=M_j+1}^{M_{j+1}} \binom{M_{i+1}}{2} - \binom{M_i}{2} - dm_i \right\} \\
&= \frac{1}{N\binom{N}{2}} \left\{ m_j \left\{ \binom{M_i+1}{2} - \binom{M_{i-1}+1}{2} \right\} - m_i \left\{ \binom{M_j+1}{2} - \binom{M_{j-1}+1}{2} \right\} \right\}.
\end{aligned}$$

(d) $i = j$

First, we calculate the probability that the first two sites in the ring are both occupied by a particle of type less than or equal to i. Let us denote this quantity by $C_i(n)$.

$$\begin{aligned}
C_i(n) &= \sum_{c=1}^{M_i} \sum_{d=1}^{M_i} E_{c,d}(N) \\
&= \sum_{c=1}^{M_i} \sum_{d=1}^{c-1} \frac{c-d}{N\binom{N}{2}} + \sum_{c=1}^{M_i-1} + \frac{1}{N^2} \frac{Ni-i^2}{N^2(N-1)} + \sum_{c=1}^{M_i-1} \sum_{d=c+2}^{M_i} \frac{1}{N^2} \\
&= \frac{\binom{M_i+1}{3}}{N\binom{N}{2}} + \frac{\binom{M_i}{2}}{N^2(N-1)} - \frac{\binom{M_i}{2}(2M_i-1)}{6N\binom{N}{2}} + \frac{\binom{M_i}{2}}{N^2} \\
&= \frac{\binom{M_i}{2}}{\binom{N}{2}}.
\end{aligned}$$

Note that $\hat{E}_{1,1}(n) = C_1(n)$ and $M_1 = m_1$, thus $\hat{E}_{1,1}(n) = \frac{\binom{m_1}{2}}{\binom{N}{2}}$ Now, let $i \in \{2, \ldots, n\}$.

We have,

$$
\begin{aligned}
\hat{E}_{i,i}(n) &= \sum_{c=M_{i-1}+1}^{M_i} \sum_{d=M_{i-1}+1}^{M_i} E_{c,d}(N) \\
&= \sum_{c=1}^{M_i}\sum_{d=1}^{M_i} E_{c,d}(N) - \sum_{c=1}^{M_{i-1}}\sum_{d=1}^{M_{i-1}} E_{c,d}(N) - \sum_{c=1}^{M_{i-1}}\sum_{d=M_{i-1}+1}^{M_i} E_{c,d}(N) - \sum_{c=M_{i-1}+1}^{M_i}\sum_{d=1}^{M_{i-1}} E_{c,d}(N) \\
&= C_i(n) - C_{i-1}(n) - \sum_{k=1}^{i-1}\left\{\hat{E}_{i,k}(n) + \hat{E}_{k,i}(n)\right\} \\
&= \frac{1}{\binom{N}{2}}\left\{\binom{M_i}{2} - \binom{M_{i-1}}{2}\right\} - \frac{m_i M_{i-1}}{N^2} - \frac{M_{i-1}(N-M_{i-1})}{N^2(N-1)} - \frac{m_i M_{i-1} M_i}{N^2(N-1)} \\
&= \frac{M_i(m_i-1)(N-M_{i-1})}{N^2(N-1)}.
\end{aligned}
$$

□

Corollary 6.4. *Let $m_c = \langle c^n \rangle$. We have for $i, j \in [n]$,*

$$
\hat{E}_{i,j}(n) = \begin{cases} \frac{2c(i-j)}{n(nc-1)}, & \textit{if } i > j, \\ \frac{1}{n^2} + \frac{i(n-i)}{n^2(nc-1)}, & \textit{if } i = j-1, \\ \frac{i(c-1)(n-i+1)}{n^2(nc-1)}, & \textit{if } i = j, \\ \frac{1}{n^2}, & \textit{if } i < j-1. \end{cases} \tag{6.2.2}
$$

Proof. We have $m_i = c$ for all $i \in [n]$. Therefore, $M_i = ic$. We get the desired result by substituting these in the formulae in Theorem 6.1. □

Remark 6.5. Note that taking $c = 1$ gives us Theorem 6.2.

Corollary 6.6. *Let ω_1, ω_2 be the first two letters in the multispecies TASEP of type m_c. The probability distribution on $\{1, \dots, n\}^2$ given by $\hat{E}_{\omega_1,\omega_2}$ converges when scaled appropriately, as $n \to \infty$, to the probability distribution on $[-1,1]^2$ whose density is given by $f_{x,y} + \mathbb{1}_{x=y} g(x)$, where $g(x) = \frac{1-x^2}{8}$ and*

$$
f_{x,y} = \begin{cases} \frac{1}{4}, & \textit{if } x > y, \\ \frac{y-x}{4}, & \textit{if } x < y. \end{cases}
$$

Proof. This result is proved using Corollary 6.4 by taking the limit $n \to \infty$ and rescaling the resulting square to $[-1,1]^2$. We take the limit in such a way that $\frac{i}{n} \to \frac{x+1}{2}$ and $\frac{j}{n} \to \frac{y+1}{2}$. We multiply the values in (6.2.2) by $n^2/4$ and take the limit $n \to \infty$. The values for two cases $i = j - 1$ and $i = j$ are, on the other hand, multiplied by $n/2$ and the measures are added to complete the proof. $\square$

Remark 6.7. Note that this is identical to the case $c = 1$ (see Corollary 4.5 of [20]).

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