Homework 1

MA 216: Graph Theory
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Indian Institute of Science

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Submit only the starred (*) problems by Aug. 22. Unless otherwise stated n is the number of vertices and m is the number of edges of the graph in the question.

1. Let $G[X, Y]$ be the bipartite graph with $\#X = r$ and $\#Y = s$. Show that $m \leq rs$ and therefore that $m \leq n^2/4$. Describe simple bipartite graphs where equality holds.
2. (*) Show that in any simple graph with at least two vertices, there exist two vertices with the same degree.
3. (*) If G is simple and $m > \binom{n-1}{2}$, show that G must be connected. For some $n > 1$, find a disconnected simple graph with $m = \binom{n-1}{2}$.
4. (*) If $m < n$, prove that G has at least $n - m$ components.
5. (*) Draw all simple labelled graphs on 3 vertices.
6. (*) Write down that adjacency matrix of K_n and compute all its eigenvalues (with multiplicities).
7. (*) If A is the adjacency matrix of G, write a simple expression for that of $\bar{G}$.
8. Show that Q_n, the n-dimensional hypercube graph, is vertex-transitive. What is $\mathrm{Aut}(Q_n)$?
9. Show that $\mathrm{Aut}(G) = \mathrm{Aut}(\bar{G})$.
10. (*) Show that if $m \geq n$, then G contains a cycle.
11. Let T be a tournament on n vertices. Show that the number of directed paths of length 2 passing through a given vertex is at most $(n-1)^2/4$.