Homework 1

MA 319: Algebraic Combinatorics
Spring 2024
Indian Institute of Science

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Submit only the starred (*) problems by Jan. 30.

1. Let $p(n)$ be the number of integer partitions of n. Prove that the generating function

$$\sum_{n\geq 0} p(n)x^n = \prod_{j\geq 1} \frac{1}{1-x^j}.$$

2. Prove that

$$H(t) := \sum_{n\geq 0} h_n(x_1, x_2, \dots)\, t^n = \prod_{j\geq 1} \frac{1}{1-x_j t} \quad \text{and} \quad E(t) := \sum_{n\geq 0} e_n(x_1, x_2, \dots)\, t^n = \prod_{j\geq 1} (1+x_j t).$$

3. Prove that

$$P(t) := \sum_{n\geq 1} p_n(x_1, x_2, \dots)\, t^{n-1} = \sum_{j\geq 1} \frac{x_j}{1-x_j t}.$$

4. Prove that

$$P(t) = \frac{d}{dt}\log H(t) \text{ and } P(-t) = \frac{d}{dt}\log E(t).$$

5. (*) Use previous exercises to show that

$$H(t) = \sum_{\lambda\in \mathrm{Par}} p_\lambda \frac{t^{|\lambda|}}{z_\lambda} \text{ and } E(t) = \sum_{\lambda\in \mathrm{Par}} p_\lambda \epsilon_\lambda \frac{t^{|\lambda|}}{z_\lambda}.$$

6. Solve Exercise 7.5, 7.7, 7.8,(*7.9) of Stanley, V2.

7. (*) Show that p_n can be written in terms of the e_j's as

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{vmatrix}.$$