Homework 2

MA 319: Algebraic Combinatorics
Spring 2024
Indian Institute of Science

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Submit only the starred (*) problems by Feb. 27.

1. Find a bijection between the partitions of n with all distinct odd parts and the self-conjugate partitions of n.

2. Let $w = w_1 w_2 \cdots w_{2n}$ be a permutation in S_{2n} such that $w_i + w_{2n+1-i} = 2n + 1$ for all i. Find the number of such permutations. Prove that the shape of $P(w)$ can be covered by n dominos.

3. For a partition λ, let λ^- (resp. λ^+) denote the set of all partitions whose Young diagram can be obtained by removing (resp. adding) one box from (resp. to) the Young diagram of λ. Find $|\lambda^+| - |\lambda^-|$.

4. (*)Let $\mathbf{Y}$ be the Young's lattice, the poset of all integer partitions ordered by inclusion of their Young diagrams. Consider the vector space $\mathbf{CY}$ of formal linear combinations of elements of $\mathbf{Y}$. Define two linear operators, D (down) and U (up) by

$$D(\lambda) = \sum_{\mu \in \lambda^-} \mu, \quad U(\lambda) = \sum_{\mu \in \lambda^+} \mu,$$

   and extend linearly.

   Prove that $DU - UD = I$, where I is the identity operator.

5. For each $\lambda \neq \emptyset$, show that $f_\lambda = \sum_{\mu \in \lambda^-} f_\mu$.

6. (*) Use Pieri's rule to show that:

$$s_{(j+1,1^k)} = \sum_{\ell=0}^{k} (-1)^\ell h_{j+\ell+1} e_{k-\ell}.$$

7. (*) For a SYT T on $[n]$, define the descent set of T as following:

$$\mathrm{Des}(T) := \{i \in [n] \mid i + 1 \text{ appear in a lower row than } i \text{ in } T\}.$$

   Let $w \in S_n$ and $\mathrm{RS}(w) = (P, Q)$. Prove than $\mathrm{Des}(w) = \mathrm{Des}(Q)$. Here $\mathrm{Des}(w)$ is the set of descents of the permutation w.