Homework 1

MA 368A: Exclusion processes
Autumn 2024
Indian Institute of Science

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Submit only the starred (*) problems by Sep. 15.

1. (*) Prove the Chapman–Kolmogorov equation for DTMCs.
2. Show that a state i in a DTMC is recurrent if and only if $P^n_{i,i} = \infty$.
3. Suppose S is finite. Then show that the total variation distance can also be written as
$$\|\mu - \nu\|_{\mathrm{TV}} = \sum_{\substack{xinS \\ \mu(x) \geq \nu(x)}} (\mu(x) - \nu(x)) .$$
4. (*) Consider a variant of the ASEP on a ring, where each particle hops with rate p_i (resp. q_i) if there are i 0's to its front (resp. back). Take $p_0 = q_0 = 0$. Show that the stationary distribution is translation invariant, but not uniform.
5. Consider a variant of the ASEP on a ring where, if a particle is at site i, it hops forward (resp. backward) with rate p_i (resp. q_i). Show that the stationary distribution is not translation invariant.
6. (*) The *ASEP with closed boundaries* is defined on a one-dimensional lattice of size L with n particles. The hopping rules in the bulk are the same as that of the ASEP on a ring. That is to say, if a particle is at site i for $2 \leq i \leq L-1$, it hops left with rate q and right with rate 1 and the hop succeeds only if the target site is empty. At the first (resp. last) site, the particle can only hop right (resp. left) with rate 1 (resp. q).
   (a) Prove that the ASEP with reflecting boundaries is irreducible.
   (b) Is the ASEP with reflecting boundaries translation-invariant?
   (c) Give an explicit formula for the stationary distribution of the ASEP with reflecting boundaries.
7. (*) Show that if $\alpha\beta = q^{L-1}\gamma\delta$, the stationary distribution of the ASEP of size L is a product measure with density $\alpha/(\alpha + q^{i-1}\gamma)$ at site i.
8. (*) Solve the system of equations for the densities in the SSEP to obtain a formula for $\langle \tau_i \rangle$ for any $i \in [L]$ and any $L \geq 1$.