ON SOME CONNECTIONS BETWEEN KOBAYASHI GEOMETRY AND PLURIPOTENTIAL THEORY

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Abstract. In this paper, we explore some connections between Kobayashi geometry and the Dirichlet problem for the complex Monge–Ampère equation. Among the results we obtain through these connections are: (i) the first theorems on the continuous extension up to ∂D of a proper holomorphic map $F : D \longrightarrow \Omega$ between domains of **differing** dimension, and (ii) a result that establishes the existence of bounded domains with "nice" boundary geometry on which Hölder regularity of the solutions to the complex Monge–Ampère equation fails. The first, a result in Kobayashi geometry, relies upon an auxiliary construction that involves solving the complex Monge–Ampère equation with Hölder estimates. The second result relies crucially on a bound for the Kobayashi metric.

1. Introduction and statement of results

This paper studies certain connections between Kobayashi geometry and the Dirichlet problem for the complex Monge–Ampère equation that are underexplored. Among the results presented in this paper are the following that are seemingly unrelated:

(1) A condition on the triple (D, Ω, F), $F : D \longrightarrow \Omega$ a proper holomorphic map between domains of **differing** dimension (in which case, nothing usually can be said about the boundary regularity of F) for F to extend continuously up to ∂D.
(2) A purely Euclidean condition that — for B-regular domains $\Omega \Subset \mathbb{C}^n$, $n \geq 2$, typically having smooth boundary — causes the failure of Hölder-regularity of the solutions to the Dirichlet problem, with smooth data, for the complex Monge–Ampère equation on Ω.

We refer the reader to Section 3.1 for the nuances of the complex Monge–Ampère equation and for some (standard) terminology associated with it. Here, we just state the fact that is essential to understanding the hypotheses of Theorems 1.1 and 1.8. Namely, if $\Omega \subsetneq \mathbb{C}^n$ is a B-regular domain — see Section 3.1 for a definition of B-regularity — then the following Dirichlet problem for the complex Monge–Ampère equation:

$$\left.\begin{aligned} \underbrace{dd^c u \wedge \dots \wedge dd^c u}_{n \text{ factors}} =: (dd^c u)^n &= f\beta_n, \quad u \in \mathcal{C}(\overline{\Omega}) \cap \mathsf{psh}(\Omega), \\ u|_{\partial\Omega} &= \varphi, \end{aligned}\right\} \tag{1.1}$$

has a unique solution for any non-negative $f \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ and any $\varphi \in \mathcal{C}(\partial\Omega; \mathbb{R})$; see [6, Theorem 4.1]. In (1.1), β_n is defined as

$$\beta_n := (i/2)^n (dz_1 \wedge d\overline{z}_1) \wedge \dots \wedge (dz_n \wedge d\overline{z}_n).$$

1.1. **The failure of Hölder regularity of solutions to the complex Monge–Ampère equation.** Our first result is motivated by the observation that very little is known one way or the other, for B-regular domains with $\mathcal{C}^2$-smooth boundaries, about Hölder regularity of the solutions to the Dirichlet problem (1.1) (even for very nice data) for domains that are **not** strongly pseudoconvex. Our result has the outcome that a bounded domain $\Omega \subsetneq \mathbb{C}^n$, $n \geq 2$, needs to satisfy just a couple of geometric conditions, which are rather easily satisfied

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simultaneously, to imply that Hölder regularity cannot hold for arbitrary data as prescribed by (1.1), even for $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ that is highly regular. Before we state this theorem, we introduce a notation needed for its statement. With Ω as above, we define (here, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$ with centre 0)

$$\mathbf{r}_\Omega(z; v) := \sup\big\{r > 0 : \big(z + (r\mathbb{D})v\big) \subset \Omega\big\} \tag{1.2}$$

for each $z \in \Omega$ and for each $v \in \mathbb{C}^n$ with $\|v\| = 1$.

The focus on B-regularity in the previous paragraph, and in the hypothesis of the theorem below, is due to the fact that we are assured of a unique solution to (1.1) when Ω is B-regular. With those words, we state our first result.

Theorem 1.1. *Let $\Omega \subsetneq \mathbb{C}^n$, $n \geq 2$, be a B-regular domain. Suppose there exist a sequence $(z_\nu)_{\nu\geq 1} \subset \Omega$, a point $\xi \in \partial\Omega$ such that $z_\nu \to \xi$, and unit vectors $\mathbf{u}_\nu \in T^{(1,0)}_{z_\nu}\Omega$, $\nu = 1, 2, 3, \dots$, such that*

$$\lim_{\nu\to\infty} \frac{\mathbf{r}_\Omega(z_\nu; \mathbf{u}_\nu)}{\big(\mathrm{dist}_{Euc}(z_\nu, \partial\Omega)\big)^\alpha} = \infty \quad \forall \alpha \in (0, 1]. \tag{1.3}$$

Then, there exist functions $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ that are restrictions of $\mathcal{C}^\infty$-smooth functions defined on neighbourhoods of $\partial\Omega$ such that, for any non-negative $f \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$, the unique solution to the Dirichlet problem (1.1) *does not belong to $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1]$.*

Remark 1.2. The proof of Theorem 1.1 will show that Kobayashi geometry reveals an obstacle for a domain to admit solutions to (1.1) of Hölder class. However, the condition (1.3) involves simple Euclidean measurements.

Remark 1.3. The reference to geometry prior to (1.2) suggests that there must be various geometric conditions implying B-regularity. This is indeed so. For Ω pseudoconvex with $\mathcal{C}^2$-smooth boundary, [31, Proposition 2.3] provides a geometric constraint on the set of weakly pseudoconvex points of $\partial\Omega$ for Ω to be B-regular. There exist geometric criteria for B-regularity for various special classes of domains: see, for instance, Proposition 4.5 and [10, Proposition 3.1].

Recalling once again the discussion prior to (1.2) on the ease with which the conditions in Theorem 1.1 simultaneously hold true: we refer the reader to Example 2.4.

Theorem 1.1 stems, in part, from a connection with Kobayashi geometry — this is one of the connections hinted at above. Its proof relies crucially on an estimate for the Kobayashi metric given by Sibony; see Result 4.4.

1.2. **Extension theorems for proper holomorphic maps.** Our principal theorem on the extension of proper holomorphic maps is motivated by the following result.

Result 1.4 (paraphrasing [13, Corollary 1.5] by Forstnerič)**.** *For each integer $m \geq 1$ there is a proper holomorphic embedding $F : \mathbb{B}^m \longrightarrow \mathbb{B}^n$, $n = m + 1 + 2s$ (where $s = s(m)$ depends only on m), such that F does not extend continuously to $\overline{\mathbb{B}^m}$.*

There is an extensive literature on the continuous extension up to ∂D, whether local or global, of proper holomorphic maps $F : D \longrightarrow \Omega$ when $D, \Omega \subsetneq \mathbb{C}^n$; i.e., when $\dim_{\mathbb{C}}(D) = \dim_{\mathbb{C}}(\Omega)$. In this case, often the geometry (resp., local geometry) of ∂D and $\partial\Omega$ suffices to ensure continuous extension of F up to ∂D (resp., locally); see, for instance, [17, 28, 9, 14, 4, 32, 2]. Result 1.4 suggests that the situation is starkly different when $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$; also see [24, 11]. It is thus natural to ask: *if $F : \mathbb{B}^m \longrightarrow \mathbb{B}^n$ is a proper holomorphic map and $m < n$, then what conditions on F would ensure that F extends continuously up to $\partial\mathbb{B}^m$?* In this setting, owing to the special geometry of Euclidean balls, the focus of research has been on seeking further information on, or to classify, such F — i.e., on so-called "rigidity properties" of F; see, for instance, [12, 8, 18, 19, 20]. Such "rigidity" theorems need to assume some *a priori* boundary regularity of F; this suggests that the natural question posed above is also a challenging one. With this as motivation, we wish to investigate — and not just for mappings between balls — what interior conditions on $F : D \longrightarrow \Omega$, F proper holomorphic and $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$, enable one to deduce continuous extension of F up to ∂D. (If $\dim_{\mathbb{C}}(D) > \dim_{\mathbb{C}}(\Omega)$, then a holomorphic map $F : D \longrightarrow \Omega$ is never proper.) Our first result addresses this problem: it covers not just the Euclidean unit balls but all pairs (D, Ω) of bounded strongly pseudoconvex domains with $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$ and presents a rather permissive interior condition on F.

Theorem 1.5. *Let $D \subsetneq \mathbb{C}^m$ and $\Omega \subsetneq \mathbb{C}^n$ be bounded strongly pseudoconvex domains with $m < n$. Let $F : D \longrightarrow \Omega$ be a proper holomorphic map, and assume that there exists some $p > m$ such that*

$$\frac{\partial F_\mu}{\partial z_j}\overline{\frac{\partial F_\nu}{\partial z_k}} \in \mathbb{L}^p(D, \mathsf{m}_{2m})$$

(*where m_{2m} denotes the $2m$-dimensional Lebesgue measure*) *for each $j, k : 1 \leq j, k \leq m$ and each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Then F extends as a continuous map $\widetilde{F} : (\overline{D}, \partial D) \longrightarrow (\overline{\Omega}, \partial\Omega)$.*

The above is, to the best of our knowledge, the first continuous-extension result for F where $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$ and is not stated for specific examples of (D, Ω). But we have a result that applies to an even larger class of pairs (D, Ω), $D \subsetneq \mathbb{C}^m$ and $\Omega \subsetneq \mathbb{C}^n$, $m, n \in \mathbb{Z}_+$, with $m < n$, of which Theorem 1.5 is a special case. We refer the reader to Section 2 for the definition of the two classes of domains that feature in the following theorem. What is more useful to see at this point is that domains of either kind are abundant and that each class contains all bounded strongly pseudoconvex domains; see [7, Section 2]. With those words, we present:

Theorem 1.6. *Let $D \subsetneq \mathbb{C}^m$ be a strongly hyperconvex Lipschitz domain, let $\Omega \subsetneq \mathbb{C}^n$ be a regular strongly hyperconvex Lipschitz domain, suppose ∂D and $\partial\Omega$ are Lipschitz manifolds, and suppose $m < n$. Let $F : D \longrightarrow \Omega$ be a proper holomorphic map, and assume that there exists some $p > m$ such that*

$$\frac{\partial F_\mu}{\partial z_j}\overline{\frac{\partial F_\nu}{\partial z_k}} \in \mathbb{L}^p(D, \mathsf{m}_{2m}). \tag{1.4}$$

for each $j, k : 1 \leq j, k \leq m$ and each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Then F extends as a continuous map $\widetilde{F} : (\overline{D}, \partial D) \longrightarrow (\overline{\Omega}, \partial\Omega)$.

We assume that readers are familiar with the notion of boundaries of open sets as Lipschitz manifolds, but we provide a definition in Section 4.

It is atypical for a pair of domains (D, Ω) with $\dim_{\mathbb{C}}(D) \leq \dim_{\mathbb{C}}(\Omega)$ to admit any proper holomorphic map $F : D \longrightarrow \Omega$. As D and Ω in Theorem 1.6 and $F : D \longrightarrow \Omega$ must satisfy several conditions, the question arises: *does there exist any pair (D, Ω) satisfying these conditions that admits a proper holomorphic map $F : D \longrightarrow \Omega$ that satisfies* (1.4)*?* (Such questions are left unanswered in a lot of the literature on the extension of proper holomorphic maps.) With some care, one can find many examples of triples (D, Ω, F), even examples where D and Ω are not strongly pseudoconvex, that satisfy all the conditions of Theorem 1.6. We present such an example in Section 2.

Our proof of Theorem 1.6 starts off on an idea of Diederich–Fornaess in [9], which involves controlling the operator norm of $F'(z)$, F as above, as z approaches ∂D. In [9] and in our proof, this relies, in part, on an estimate for the Kobayashi metric on Ω. Thus, Theorem 1.6 lies in the realm of Kobayashi geometry. Unlike in [9], $\partial\Omega$ can be quite rough, but our conditions on Ω enable us to appeal to results on the regularity of the solutions to the Dirichlet problem (1.1) to estimate the Kobayashi metric on Ω. This new approach is the content of Proposition 3.4 below. But the chief novelty of Theorem 1.6 lies in dealing with the condition $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$. In many of the works cited above in which $\dim_{\mathbb{C}}(D) = \dim_{\mathbb{C}}(\Omega)$, the above-mentioned control of $\|F'(\cdot)\|$ also involves the use of a Hopf-type lemma applied to an auxiliary plurisubharmonic function on D. This function is not definable when $\dim_{\mathbb{C}}(D) < \dim_{\mathbb{C}}(\Omega)$, and the construction of a suitable substitute thereof requires a second, and careful, appeal to a complex Monge–Ampère equation wherein the datum $f \geq 0$ is far less regular than in (1.1) (see Step 2 of the content of Section 6). This is yet another of the connections alluded to at the top of this section.

Whether there exist any proper holomorphic maps from D to Ω is no longer an issue when $D = \mathbb{D}$ and Ω is a bounded convex domain. This observation is due to Lempert when Ω is strictly convex and has $\mathcal{C}^3$-smooth boundary [23] and to Royden–Wong in the general case [29]. For Ω as in either of these works, they establish that given any two distinct points $w_1, w_2 \in \Omega$, there exists a complex geodesic $\psi : \mathbb{D} \longrightarrow \Omega$ such that $w_1, w_2 \in \psi(\mathbb{D})$. A holomorphic map $\psi : \mathbb{D} \longrightarrow \Omega$ is called a *complex geodesic* if it is an isometry for the Kobayashi distances on $\mathbb{D}$ and Ω. Clearly, any complex geodesic is a proper holomorphic map. Whether a complex geodesic extends as a continuous map on $\overline{\mathbb{D}}$ is a subtle question. This question was first considered in [23] and answered in the affirmative when Ω is as in [23] and is strongly convex. To go beyond the strongly convex case, one needs the following

Definition 1.7. A convex domain $\Omega \subsetneq \mathbb{C}^n$ is said to be $\mathbb{C}$-*strictly convex* if, for each $p \in \partial\Omega$, there exists a support hyperplane of Ω containing p—denote it as $\mathcal{H}_p$—such that the $\mathbb{C}$-affine hyperplane

$$\widetilde{\mathcal{H}}_p := p + \big((\mathcal{H}_p - p) \cap i(\mathcal{H}_p - p)\big)$$

satisfies $\widetilde{\mathcal{H}}_p \cap \overline{\Omega} = \{p\}$.

The above definition is relevant because of an example by Bharali of a bounded convex domain with $\mathcal{C}^\infty$-smooth boundary that admits complex geodesics that do **not** extend as continuous maps on $\overline{\mathbb{D}}$ [5, Example 1.2]. The domain in this example is not $\mathbb{C}$-strictly convex.

For $\mathbb{C}$-strictly convex domains, there have been several recent results establishing the continuous extension of complex geodesics up to $\partial\mathbb{D}$; see [5, 33, 25]. In all these works, with Ω as in Definition 1.7, Ω is either assumed to have boundary that is strictly more regular than $\mathcal{C}^1$ or, when $\partial\Omega$ is $\mathcal{C}^1$-smooth, to satisfy a condition stronger than $\mathbb{C}$-strict convexity. We follow an approach different from these works to establish a continuous-extension result where Ω is **not** assumed to have $\mathcal{C}^1$-smooth boundary.

Theorem 1.8. *Let $\Omega \subsetneq \mathbb{C}^n$ be a bounded $\mathbb{C}$-strictly convex domain.*

- (a) *Then, Ω is B-regular.*
- (b) *Assume that the canonical function for Ω admits a modulus of continuity ω such that $\sqrt{\omega}$ satisfies a Dini condition. Then, every complex geodesic of Ω extends as a continuous map on $\overline{\mathbb{D}}$.*

To prove the above theorem, we appeal to some of the methods used in proving Theorem 1.6 (which reduces our proof to an appeal to a Hardy–Littlewood-type result). This is why we need to establish part (a) above. Theorem 1.8-(a), among other things, ensures that the following definition makes sense.

Definition 1.9. Let Ω be a B-regular domain. The *canonical function* of Ω is the unique solution to the Dirichlet problem (1.1) with $f \equiv 0$ and $\varphi := -2\|\cdot\|^2\big|_{\partial\Omega}$.

Since we suggested that a novelty of Theorem 1.8 is that Ω therein is not assumed to have $\mathcal{C}^1$-smooth boundary, the question arises: *do there exist convex domains $\Omega \subsetneq \mathbb{C}^n$ such that $\partial\Omega$ is not $\mathcal{C}^1$-smooth and satisfy the hypothesis of Theorem 1.8?* The following corollary of Theorem 1.8 indirectly (see its proof in Section 7) answers this question in the affirmative.

Corollary 1.10. *Let Ω be a non-empty finite intersection of bounded strongly convex domains in $\mathbb{C}^n$. Then, every complex geodesic of Ω extends as a continuous map on $\overline{\mathbb{D}}$.*

2. Two examples

This section is devoted to the examples mentioned in Section 1. But first, let us fix some notation that will be used right below and in later sections.

2.1. **Basic notation.** The following is a list of recurring notations (some of which were used without comment in the previous section). In our list, Ω denotes a domain in $\mathbb{C}^n$.

1. For $v \in \mathbb{C}^n$, $\|v\|$ will denote the Euclidean norm of v.
2. Given a non-empty set $S \subseteq \mathbb{C}^n$, the class of all **real**-valued functions that satisfy a Hölder condition with Hölder exponent α, $\alpha \in (0,1]$ (to clarify: a Hölder condition with Hölder exponent $= 1$ implies a Lipschitz condition) will be denoted by $\mathcal{C}^{0,\alpha}(S)$.
3. $\mathcal{C}(\overline{\Omega}) \cap \mathsf{psh}(\Omega)$ will denote the class of real-valued functions $u : \overline{\Omega} \longrightarrow \mathbb{R}$ that are continuous and such that $u|_\Omega$ is plurisubharmonic.
4. For $\Omega \subsetneq \mathbb{C}^n$, given a point $z \in \Omega$, $\delta_\Omega(z)$ will denote
$$\mathrm{dist}_{Euc}(z, \partial\Omega) := \inf\{\|z - \xi\| : \xi \in \partial\Omega\}$$
5. k_Ω will denote the Kobayashi (pseudo)metric on Ω.
6. Given a point $x \in \mathbb{R}^d$ and $r > 0$, $B^d(x, r)$ will denote the open Euclidean ball in $\mathbb{R}^d$ with radius r and centre x.
7. Given a point $z \in \mathbb{C}^n$ and $r > 0$, $\mathbb{B}^n(z, r)$ will denote the open Euclidean ball in $\mathbb{C}^n$ with radius r and centre z. For simplicity, we write $\mathbb{B}^n := \mathbb{B}^n(0,1)$ and $\mathbb{D} := \mathbb{B}^1(0,1)$ (which already appear in Section 1).

A final clarification: on several occasions, we will work with $\mathbb{C}$-valued $\mathbb{L}^\infty$-functions on some open set $\mathscr{O} \subseteq \mathbb{C}^n$ — i.e., functions that are essentially bounded with respect to the Lebesgue measure. For simplicity of notation, we will denote this class by $\mathbb{L}^\infty(\mathscr{O})$.

2.2. **Examples.** We will first present the example alluded to prior to (1.2) and right after Remark 1.3. To this end, we will need the following result:

Result 2.1 (Graham, [15])**.** *Let Ω be a bounded convex domain in $\mathbb{C}^n$. Then:*
$$\frac{\|v\|}{2\mathbf{r}_\Omega(z; v/\|v\|)} \leq k_\Omega(z; v) \leq \frac{\|v\|}{\mathbf{r}_\Omega(z; v/\|v\|)} \quad \forall z \in \Omega \ \text{and} \ \forall v \in \mathbb{C}^n \setminus \{0\}.$$

The following proposition will be needed in discussing Example 2.4.

Proposition 2.2. *Let $\varphi : (0, +\infty) \longrightarrow (0, +\infty)$ be a function of class $\mathcal{C}^k$, $k \geq 2$, such that*

(a) $\varphi^{(j)}$ *extends continuously to* 0 *for* $0 \leq j \leq k$.
(b) $\lim_{x\to 0^+}(\varphi^{(j-1)}(x) - \varphi^{(j-1)}(0))/x = \varphi^{(j)}(0)$ *for* $1 \leq j \leq k$.
(c) $\varphi^{(j)}(0) = 0$ *for* $0 \leq j \leq s$ *for some* $s : 1 \leq s \leq k$.
(d) φ *is strictly increasing and convex.*
(e) $\varphi(a) = 1$ *for some* $a > 0$ *and the behaviour of* φ *near* a *is such that*
$$\Omega_\varphi := \Big\{z \in \mathbb{C}^n : \varphi(z_1\overline{z}_1) + \sum_{2\leq j\leq n} |z_j|^2 < 1\Big\}$$
is a convex domain with $\mathcal{C}^k$*-smooth boundary.*

Then, Ω_φ is B-regular. Fix $l = 2, \dots, n$. For each $\varepsilon \in (-1, -1/2)$, write
$$z_\varepsilon := (\underbrace{0, \dots, 0}_{(l-1)\ \text{entries}}, \varepsilon, 0, \dots, 0).$$
Let $\boldsymbol{\epsilon}_1 := (1, 0, \dots, 0)$. Then, for every $\varepsilon \in (-1, -1/2)$:

(i) $\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2} \leq \mathbf{r}_{\Omega_\varphi}(z_\varepsilon; \boldsymbol{\epsilon}_1) \leq 2\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}$.

(*ii*) *Furthermore, we have the estimate*

$$\frac{\|v\|}{4\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}} \leq k_{\Omega_\varphi}(z_\varepsilon; v) \leq \frac{\|v\|}{\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}} \quad \forall v \in \text{span}_{\mathbb{C}}\{\boldsymbol{\epsilon}_1\}.$$

Remark 2.3. The fullest strength of condition (c) above is not used in our proof. The assertion about $\varphi^{(s)}(0) = 0$ and $s \geq 1$ ensures that $T_\xi\partial\Omega_\varphi$ has order of contact with $\partial\Omega_\varphi$ at ξ greater than 2, for any ξ of the form $(0, \xi_2, \dots, \xi_n)$ such that $(\xi_2, \dots, \xi_n) \in \mathbb{S}^{n-1}$, the unit sphere in $\mathbb{C}^{n-1}$. In other words, condition (c) states that Ω_φ is weakly pseudoconvex.

We now provide

The proof of Proposition 2.2. First we will prove that Ω_φ is B-regular. If we were to appeal to Proposition 4.5, we would have to compute $T_\xi\partial\Omega_\varphi$ for each $\xi \in \partial\Omega_\varphi$, which is somewhat unpleasant. Instead, since Ω_φ is a bounded, convex, Reinhardt domain in $\mathbb{C}^n$, by [10, Proposition 3.1] it is enough to show that Ω_φ does not have any non-constant analytic disc in $\partial\Omega_\varphi$. If possible, let $\Psi : \mathbb{D} \longrightarrow \mathbb{C}^n$ be a non-constant holomorphic map such that $\Psi(\mathbb{D}) \subseteq \partial\Omega_\varphi$. Let $\Psi := (\psi_1, \dots, \psi_n)$, where $\psi_i : \mathbb{D} \longrightarrow \mathbb{C}$ are holomorphic maps for $1 \leq i \leq n$. Since $\Psi(\mathbb{D}) \subseteq \partial\Omega_\varphi$, for every $\zeta \in \mathbb{D}$ we have

$$\varphi\big(\psi_1(\zeta)\overline{\psi_1(\zeta)}\big) + \sum_{j=2}^{n} |\psi_j(\zeta)|^2 = 1.$$

Applying the operator $\partial^2/\partial\zeta\partial\overline{\zeta}$ to both sides we get

$$\varphi''(|\psi_1|^2)\left|\overline{\frac{\partial\psi_1}{\partial\zeta}}\psi_1\right|^2 + \varphi'(|\psi_1|^2)\left|\frac{\partial\psi_1}{\partial\zeta}\right|^2 + \sum_{j=2}^{n}\left|\frac{\partial\psi_j}{\partial\zeta}\right|^2 \equiv 0 \quad \text{on } \mathbb{D}. \tag{2.1}$$

From conditions (a) and (d) it follows that $\varphi'(x) \geq 0$ and $\varphi''(x) \geq 0$ for all $x \geq 0$. Therefore, from (2.1), we have

$$\frac{\partial\psi_j}{\partial\zeta} \equiv 0 \quad \forall j = 2, \dots, n.$$

Therefore, ψ_j is constant for every $j = 2, \dots, n$. Since Ψ is non-constant, ψ_1 must be non-constant. In particular, $\psi_1 \not\equiv 0$ on $\mathbb{D}$. Thus, there exists a non-empty open set $U \subseteq \mathbb{D}$ such that $|\psi_1(\zeta)| > 0$ for all $\zeta \in U$. From (2.1), and as $\varphi'(x) > 0$ for $x > 0$,

$$\left|\frac{\partial\psi_1}{\partial\zeta}\right| \equiv 0 \quad \text{on } U.$$

Therefore, ψ_1 is constant on U. Hence, by the Identity Principle, ψ_1 is constant on $\mathbb{D}$, which contradicts the fact that Ψ is non-constant. Hence, Ω_φ is B-regular.

Fix $l \in \{2, \dots, n\}$. Let $\varepsilon \in (-1, -1/2)$. Hence, $\delta_{\Omega_\varphi}(z_\varepsilon) = 1 + \varepsilon$. Fix $\lambda \in \mathbb{D}$. Define $w_\lambda := z_\varepsilon + \big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}\lambda\boldsymbol{\epsilon}_1 = \big((\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon)))^{1/2}\lambda, 0, \dots, 0, \varepsilon, 0, \dots, 0\big)$. Since φ is convex and $\varphi(0) = 0$, it follows that

$$\begin{aligned}\varphi\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))|\lambda|^2\big) + \varepsilon^2 &\leq |\lambda|^2\varphi\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big) + \varepsilon^2 \\ &\leq \delta_{\Omega_\varphi}(z_\varepsilon) + \varepsilon^2 \leq 1 + \varepsilon + \varepsilon^2 < 1.\end{aligned}$$

Therefore, $w_\lambda \in \Omega_\varphi$ for all $\lambda \in \mathbb{D}$. Hence, for all $\varepsilon \in (-1, -1/2)$ it follows by definition that

$$\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2} \leq \mathbf{r}_{\Omega_\varphi}(z_\varepsilon; \boldsymbol{\epsilon}_1). \tag{2.2}$$

Now, fix $\lambda = 1/\sqrt{2}$ and consider the point

$$\widehat{w} := z_\varepsilon + \big(2\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}(1/\sqrt{2})\boldsymbol{\epsilon}_1 = \big((\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon)))^{1/2}, 0, \dots, 0, \varepsilon, 0, \dots, 0\big).$$

We compute:

$$\varphi\big(\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon))\big) + \varepsilon^2 = 2\delta_{\Omega_\varphi}(z_\varepsilon) + \varepsilon^2$$
$$= 2(1+\varepsilon) + \varepsilon^2 = 1 + (1+\varepsilon)^2 > 1$$

By our definition of Ω_φ, $z_\varepsilon + \big(2\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}(1/\sqrt{2})\boldsymbol{\epsilon}_1 \notin \Omega_\varphi$. Therefore, by definition we have $\mathbf{r}_{\Omega_\varphi}(z_\varepsilon; \boldsymbol{\epsilon}_1) \leq \big(2\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}$. By condition (d), φ^{-1} is concave. Hence, from the fact that $\varphi(0) = 0$, it follows that $\varphi^{-1}(2t) \leq 2\varphi^{-1}(t)$ for all $t \geq 0$. Therefore, for all $\varepsilon \in (-1, -1/2)$

$$\mathbf{r}_{\Omega_\varphi}(z_\varepsilon; \boldsymbol{\epsilon}_1) \leq \big(2\varphi^{-1}(2\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2} \leq 2\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}. \tag{2.3}$$

Hence, combining (2.2) and (2.3), part (i) follows.

Therefore, by Result 2.1, for every $\varepsilon \in (-1, -1/2)$ we have

$$\frac{\|v\|}{4\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}} \leq k_{\Omega_\varphi}(z_\varepsilon; v) \leq \frac{\|v\|}{\big(\varphi^{-1}(\delta_{\Omega_\varphi}(z_\varepsilon))\big)^{1/2}} \quad \forall v \in \operatorname{span}_{\mathbb{C}}\{\boldsymbol{\epsilon}_1\}.$$

□

Having established the above proposition, we can present a family of domains that illustrate how abundant domains satisfying the conditions of Theorem 1.1 are. This substantiates the discussion right after Remark 1.3.

Example 2.4. *An example demonstrating that there exists a sequence of domains Ω_n, $n \geq 2$, satisfying all the conditions stated in Theorem 1.1.*

Let us define the function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ as

$$\varphi(x) := \begin{cases} 0, & \text{if } x = 0, \\ e^2 \exp(-1/x)/3, & \text{if } 0 < x < 1/2, \\ (4x-1)/3, & \text{otherwise.} \end{cases}$$

The conditions (a)–(c) in Proposition 2.2, with $k = 2$ and $s = 2$, follow from elementary computations and that φ is strictly increasing is almost self-evident. Note that the function $x \longmapsto e^2 \exp(-1/x)/3$ is not convex on the whole of $(0, +\infty)$. One restricts the latter function to $(0, 1/2)$ to ensure convexity of φ. Now, we will show that φ is convex on $(0, +\infty)$. It is easy to check that φ is $\mathcal{C}^2$ on $(0, +\infty)$. A computation gives

$$\varphi''(x) = \begin{cases} e^2 \exp(-1/x)(1-2x)/3x^4, & \text{if } 0 < x < 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that $\varphi'' \geq 0$, and hence, φ is convex.

Now, set $\Omega_n := \Big\{z \in \mathbb{C}^n : \varphi(z_1\overline{z}_1) + \sum_{2 \leq j \leq n} |z_j|^2 < 1\Big\}$. Since $\varphi(1) = 1$, we have $(1, 0, \dots, 0) \in \partial\Omega_n$. From the definition of φ, it follows that in a neighbourhood of $(1, 0, \dots, 0)$, $\partial\Omega_n$ is given by

$$|z_1| = \sqrt{1 - \Big(3 \sum_{2 \leq j \leq n} |z_j|^2/4\Big)}.$$

Therefore, $\partial\Omega_n$ is $\mathcal{C}^\infty$-smooth in a neighbourhood of $(1, 0, \dots, 0)$. Since Ω_n is Reinhardt, the above holds true in a neighbourhood of any $(e^{i\theta}, 0, \dots, 0) \in \partial\Omega_n$, where $\theta \in [0, 2\pi)$. This, together with the conclusions of the previous paragraph implies that $\partial\Omega_n$ is $\mathcal{C}^2$-smooth. Writing $z_1 = x_1 + iy_1$, the real Hessian of $z_1 \longmapsto \varphi(z_1\overline{z}_1)$ is

$$\begin{bmatrix} 4x_1^2\varphi''(z_1\overline{z}_1) + 2\varphi'(z_1\overline{z}_1) & 4x_1y_1\varphi''(z_1\overline{z}_1) \\ 4x_1y_1\varphi''(z_1\overline{z}_1) & 4y_1^2\varphi''(z_1\overline{z}_1) + 2\varphi'(z_1\overline{z}_1) \end{bmatrix}.$$

Thus, the real Hessian of the function $z \longmapsto \varphi(z_1\overline{z}_1) + \sum_{2\leq j\leq n}|z_j|^2$ is positive semi-definite, due to which the second fundamental form of $\partial\Omega$ is non-negative at each $\xi \in \partial\Omega_n$. Thus, the condition (e) holds true as well (with $a = 1$).

Since all the conditions stated in Proposition 2.2 are satisfied, we need to establish one last condition. Let

$$z_\nu := (0, -1 + 1/(\nu + 2), 0, \dots, 0) \quad \text{and} \quad \mathbf{u}_\nu := \boldsymbol{\epsilon}_1,$$

for $\nu = 1, 2, 3, \dots$. By Proposition 2.2, we have

$$\mathbf{r}_{\Omega_n}(z_\nu; \mathbf{u}_\nu) \geq \frac{1}{\big(\log(e^2/3\delta_{\Omega_n}(z_\nu))\big)^{1/2}}$$

for $\nu = 1, 2, 3, \dots$; clearly, (1.3) holds true. Therefore, Ω_n satisfies all the conditions stated in Theorem 1.1. ◀

We now provide an example that relates to the discussion right after Theorem 1.6. In particular, our example demonstrates that Theorem 1.6 is **not** vacuously true. This is therefore an appropriate place to present a pair of definitions that were deferred to a later section in our discussion in Section 1.2.

Definition 2.5. Let $\Omega \subsetneq \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$.

- (a) (Charabati, [7]) We say that Ω is a *strongly hyperconvex Lipschitz domain* if there exists a neighbourhood U of $\overline{\Omega}$ and a Lipschitz plurisubharmonic function $\rho : U \longrightarrow \mathbb{R}$ such that:
  - (1) $\rho < 0$ in Ω and $\partial\Omega = \{z \in U : \rho(z) = 0\}$.
  - (2) There exists a constant $c > 0$ such that $dd^c\rho \geq c\omega_n$ on Ω in the sense of currents.

  Here, ω_n denotes the standard Kähler form, $(i/2)\sum_{j=1}^n dz_j \wedge d\overline{z}_j$, on $\mathbb{C}^n$.
- (b) We say that Ω is a *regular strongly hyperconvex Lipschitz domain* if Ω is a strongly hyperconvex Lipschitz domain and if, writing $dd^c\rho = \sum_{\mu,\nu=1}^n b_{\mu\overline{\nu}}\, dz_\mu \wedge d\overline{z}_\nu$, each $b_{\mu\overline{\nu}} \in \mathbb{L}^\infty(\Omega)$.

Example 2.6. *An example demonstrating that there exist domains $D \subsetneq \mathbb{C}^m$, $\Omega \subsetneq \mathbb{C}^n$, $m < n$, and a proper holomorphic map $F : D \longrightarrow \Omega$ such that*

- *D, Ω, and F satisfy the conditions of Theorem 1.6,*
- *D, Ω have non-smooth boundaries.*

Express $\zeta \in \mathbb{C}$ or z_2, where $(z_1, z_2) \in \mathbb{C}^2$, as $u + iv$, $u, v \in \mathbb{R}$, and define $h(\zeta) = 2u^2 - \beta(v) + v^4$, where

$$\beta(v) := \begin{cases} v^2, & \text{if } v \geq 0, \\ -v^2, & \text{if } v < 0. \end{cases}$$

Therefore,

$$h(\zeta) = 2u^2 + (v^2 - 1/2)^2 - 1/4 \;\text{ whenever } v \geq 0, \tag{2.4}$$

$$h(\zeta) \geq -1/4 \quad \forall \zeta \in \mathbb{C}. \tag{2.5}$$

Let us define

$$D := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + h(z_2) < 0\},$$
$$\Omega := \{(w_1, w_2, w_3) \in \phi(D) \times \mathbb{C} : |w_1^2 - 1|^2 + h(w_2) + |w_3|^2 < 0\},$$

where $\phi(z_1, z_2) := (\sqrt{z_1 + 1}, z_2)$, $(z_1, z_2) \in D$, and where $\sqrt{\cdot}$ denotes the principal branch of the square root. We need to argue that $\phi \in \mathcal{O}(D; \mathbb{C}^2)$. We will show that D is bounded, a consequence of which will be that $\phi \in \mathcal{O}(D; \mathbb{C}^2)$. Define the continuous functions $\rho_1 : \mathbb{C}^2 \longrightarrow \mathbb{R}$ and $\rho_2 : \mathbb{C}^3 \longrightarrow \mathbb{R}$ by

$$\rho_1(z_1, z_2) = |z_1|^2 + h(z_2), \quad \rho_2(z_1, z_2, z_3) = |z_1^2 - 1|^2 + h(z_2) + |z_3|^2.$$

By definition, $(z_1, z_2) \in \overline{D}$ implies that $\mathsf{Im}(z_2) \not< 0$. Therefore, $D \subseteq \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + 2(\mathsf{Re}(z_2))^2 + ((\mathsf{Im}(z_2))^2 - 1/2)^2 < 1/4\}$ and, hence, D is a bounded domain in $\mathbb{C}^2$. Moreover, $(z_1, z_2) \in D$ implies that $|z_1| < 1/2$. Therefore, ϕ is holomorphic on D.

It is easy to compute that for every $(u, v) \in \mathbb{R}^2$,

$$\frac{\partial^2 h}{\partial u^2}(u, v) = 4 \quad \text{and} \quad \frac{\partial^2 h}{\partial v^2}(u, v) = \begin{cases} -2 + 12v^2, & \text{if } v \geq 0, \\ 2 + 12v^2, & \text{if } v < 0. \end{cases} \tag{2.6}$$

Note that $\rho_1|_{\{(z_1,z_2)\,:\,\mathsf{Im}(z_2)>0\}}$ and $\rho_1|_{\{(z_1,z_2)\,:\,\mathsf{Im}(z_2)<0\}}$ are of class $\mathcal{C}^2$. From this and from (2.6),

$$h|_{\{u+iv\,:\,v<0\}} \in \mathsf{sh}(\{u + iv : v < 0\}) \quad \text{and} \quad h|_{\{u+iv\,:\,v>0\}} \in \mathsf{sh}(\{u + iv : v > 0\}).$$

Thus, if we fix $u_0 + iv_0 \in \mathbb{C}$ such that $v_0 \neq 0$, then the averages of h on circles with centre $u_0 + iv_0$ and radii in $(0, |v_0|)$ dominate $h(u_0 + iv_0)$. Now observe that

$$\frac{1}{2\pi}\int_0^{2\pi} h(u_0 + re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \big(2(u_0 + r\cos\theta)^2 + (r\sin\theta)^4\big)\, d\theta \geq 2u_0^2 = h(u_0)$$

for each $u_0 \in \mathbb{R}$ and $r > 0$. From the last two observations, we conclude that $h \in \mathsf{sh}(\mathbb{C})$. From the above discussion, we establish three of the conditions for D to be a strongly hyperconvex Lipschitz domain, the third being a consequence of our last assertion (about h) and the definition of ρ_1:

- ρ_1 is (uniformly) Lipschitz on any relatively compact domain in $\mathbb{C}^2$.
- $\partial D = \rho_1^{-1}\{0\}$.
- $\rho_1 \in \mathsf{psh}(\mathbb{C}^2)$.

Moreover, as $(z_1, z_2) \in D$ implies that $\mathsf{Im}(z_2) > 0$, (2.6) implies, after an easy calculation, that:

- For every $(z_1, z_2) \in D$,

$$dd^c\rho_1(z_1, z_2) = (i/2)\,\big(dz_1 \wedge d\overline{z}_1 + \big(1/2 + 3(\mathsf{Im}(z_2))^2\big)dz_2 \wedge d\overline{z}_2\big) \geq (1/2)\omega_2(z_1, z_2).$$

It remains to show that D is connected. To this end, observe that, by the definition of D, it suffices to show that

$$\Delta := \{(u, v) \in \mathbb{R}^2 : h(u, v) < 0\}$$

is connected. By the definition of h, if $(u, v) \in \partial\Delta$, then $v \geq 0$. Thus, by (2.4), $(u, v) \in \partial\Delta$ implies that

$$\begin{aligned} &\big(\sqrt{2}u, v^2 - (1/2)\big) \text{ lies on the circle with centre 0 and radius } 1/2 \\ \iff &\big(2\sqrt{2}u, 2(v^2 - (1/2))\big) \text{ lies on the circle with centre 0 and radius } 1. \end{aligned}$$

Thus, we have the continuous, bijective map $\psi : \mathbb{S}^1 \longrightarrow \partial\Delta$:

$$\psi(s, t) := \left(\frac{s}{2\sqrt{2}}, \sqrt{\frac{t+1}{2}}\right), \quad (s, t) \in \mathbb{S}^1.$$

As $\mathbb{S}^1$ is compact, ψ is a homeomorphism; thus $\partial\Delta$ is a Jordan curve. It follows from the Jordan Curve Theorem that Δ is connected. By our remarks preceding the definition of Δ, we conclude that D is a domain. Together with the four bullet-points earlier in this paragraph, it follows that D is a strongly hyperconvex Lipschitz domain.

It is elementry to see that the only point $p \in \mathbb{C}^2$ such that $\nabla\rho_1(p) = 0$ and $p \in \partial D$ is $p = (0, 0)$. Therefore, D has a non-smooth boundary. Since ∂D is $\mathcal{C}^\infty$-smooth in a small neighbourhood of ξ for each $\xi \in \partial D \setminus \{(0, 0)\}$, we only need to examine ∂D around $(0, 0)$. Recall that if $(z_1, z_2) \in \overline{D}$, for $z_2 = u + iv$, then $v \geq 0$. Thus, we will solve the equation

$$|z_1|^2 + 2u^2 - v^2 + v^4 = 0, \quad \text{and}$$

$$v \geq 0,$$

explicitly for v in terms of z_1 and u for $\|(z_1, u)\|$ small. Solving the auxiliary quadratic equation

$$X^2 - X + (2u^2 + |z_1|^2) = 0,$$

we find that there exists a $\delta > 0$ sufficiently small that, if Γ_δ denotes the connected component of $\partial D \cap B^3(0,\delta) \times \mathbb{R}$ containing $(0,0)$, then

$$\Gamma_\delta : v = \sqrt{|z_1|^2 + 2u^2} + O(\|(z_1, u)\|^2) \quad \forall (z_1, u) \in B^3(0,\delta).$$

By our previous remarks about the points in $\partial D \setminus \{(0,0)\}$, ∂D is a Lipschitz manifold. *We have shown that D satisfies all the conditions of Theorem 1.6.*

Since D is connected and ϕ is continuous, Ω is connected. By definition, $(w_1, w_2, w_3) \in \overline{\Omega}$ implies that $\mathsf{Im}(w_2) \not< 0$. Hence, $\Omega \subseteq \{(w_1, w_2, w_3) \in \mathbb{C}^3 : |w_1^2 - 1|^2 + 2(\mathsf{Re}(w_2))^2 + ((\mathsf{Im}(w_2))^2 - 1/2)^2 + |w_3|^2 < 1/4\}$. Therefore, Ω is a bounded domain in $\mathbb{C}^3$. We have shown that $h \in \mathsf{sh}(\mathbb{C})$; by definition it follows that $\rho_2 \in \mathsf{psh}(\mathbb{C}^3)$. Hence, from the above discussion, and a few easy estimates, we have:

- ρ_2 is (uniformly) Lipschitz on any relatively compact domain in $\mathbb{C}^3$.
- For every $(w_1, w_2, w_3) \in \Omega$,

$$\begin{aligned} &dd^c\rho_2(w_1, w_2, w_3) \\ &\quad = (i/2)\left(4|w_1|^2 dw_1 \wedge d\overline{w}_1 + \left(1/2 + 3(\mathsf{Im}(w_2))^2\right) dw_2 \wedge d\overline{w}_2 + dw_3 \wedge d\overline{w}_3\right) \\ &\quad \geq (1/2)\omega_3(w_1, w_2, w_3). \end{aligned} \tag{2.7}$$

Therefore, since Ω is bounded, $dd^c\rho_2|_\Omega$ is a $(1,1)$-current with $\mathbb{L}^\infty(\Omega)$-coefficients. Now, define

$$U := \text{the connected component of } \{(w_1, w_2, w_3) \in \mathbb{C}^3 : \rho_2(w_1, w_2, w_3) < 1/2\} \text{ containing } \overline{\Omega}.$$

Note that if $(w_1, w_2, w_3) \in \rho_2^{-1}\{0\}$, then $(-w_1, w_2, w_3) \in \rho_2^{-1}\{0\}$. However, we claim that $(\rho_2|_U)^{-1}\{0\} = \partial\Omega$. To see this: note that by (2.5), $\rho_2(w_1, w_2, w_3) < 1/2$ implies that $|w_1|^2 \geq 1 - |w_1^2 - 1| \geq (1 - \sqrt{3}/2) > 0$. Therefore,

$$\{(0, w_2, w_3) : (w_2, w_3) \in \mathbb{C}^2\} \cap \overline{U} = \emptyset. \tag{2.8}$$

From (2.8), it easily follows that $\left(\rho_2^{-1}\{0\} \cap U, \rho_2^{-1}\{0\} \cap (\mathbb{C}^3 \setminus U)\right)$ is a separation of $\rho_2^{-1}\{0\}$. Therefore:

- $(\rho_2|_U)^{-1}\{0\} = \partial\Omega$.

From the three bullet-points in this paragraph, we conclude that Ω is a regular strongly hyperconvex Lipschitz domain.

Again, it is elementry to see that the only point $q \in \mathbb{C}^3$ such that $\nabla\rho_2(q) = 0$ and $q \in \partial\Omega$ is $q = (1,0,0)$. Therefore, Ω has a non-smooth boundary and in a manner similar to our previous argument for D, we can conclude that $\partial\Omega$ is a Lipschitz manifold. *We have now shown that Ω satisfies all the conditions of Theorem 1.6.*

Finally, let us define

$$F(z_1, z_2) := \left(\sqrt{z_1 + 1}, z_2, 0\right) \quad \forall (z_1, z_2) \in D.$$

Since ϕ is holomorphic, $F : D \longrightarrow \Omega$ is holomorphic. Now, to prove that $F : D \longrightarrow \Omega$ is proper, it suffices to show that $\phi = (\phi_1, \phi_2) : D \longrightarrow \phi(D)$ is proper. To this end, consider $\left((z_{1,\nu}, z_{2,\nu})\right)_{\nu \geq 1} \subset D$ such that $(z_{1,\nu}, z_{2,\nu}) \longrightarrow \xi$ for some $\xi \in \partial D$. This implies that

$$\begin{aligned} &|z_{1,\nu}|^2 + h(z_{2,\nu}) \longrightarrow 0 \\ \implies &|(\phi_1(z_{1,\nu}, z_{2,\nu}))^2 - 1|^2 + h(\phi_2(z_{1,\nu}, z_{2,\nu})) \longrightarrow 0, \end{aligned}$$

as $\nu \to \infty$. Therefore, since $\phi(D) \subseteq \{(w_1, w_2) \in \mathbb{C}^2 : |w_1^2 - 1|^2 + h(w_2) < 0\}$, it follows that $\big(\phi(z_{1,\nu}, z_{2,\nu})\big)_{\nu\geq 1}$ exits every compact in $\phi(D)$. Hence, $\phi : D \longrightarrow \phi(D)$ is proper. Let $F = (F_1, F_2, F_3)$. Since $(z_1, z_2) \in D$ implies that $|z_1| < 1/2$, it follows that $\frac{\partial F_1}{\partial z_1} = \frac{1}{2\sqrt{z_1+1}} \in \mathbb{L}^\infty(D)$. Therefore, clearly

$$\frac{\partial F_\mu}{\partial z_j}\,\overline{\frac{\partial F_\nu}{\partial z_k}} \in \mathbb{L}^\infty(D)$$

for each $j, k : 1 \leq j, k \leq 2$ and each $\mu, \nu : 1 \leq \mu, \nu \leq 3$. Hence, D, Ω, and F satisfy the conditions of Theorem 1.6 such that D and Ω have non-smooth boundaries. ◀

3. Analytical preliminaries

This section is devoted to several important observations and results related, primarily, to the proofs of the theorems in Section 1.2.

3.1. **Concerning the complex Monge–Ampère equation.** We begin with a brief discussion on B-regular domains, introduced in Section 1 and whose definition was deferred to this section.

Definition 3.1. Let Ω be a bounded domain in $\mathbb{C}^n$. We say that Ω is B-regular if $\partial\Omega$ is a B-regular set: i.e., $\partial\Omega$ has the property that each function $\varphi \in \mathcal{C}(\partial\Omega; \mathbb{R})$ is the uniform limit on $\partial\Omega$ of a sequence $(u_\nu)_{\nu\geq 1}$ of continuous plurisubharmonic functions defined on open neighbourhoods of $\partial\Omega$ (each such neighbourhood depending on the function u_ν).

As mentioned in Section 1, [6, Theorem 4.1] by Błocki establishes that the Dirichlet problem (1.1) admits a unique solution for any non-negative $f \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ and any $\varphi \in \mathcal{C}(\partial\Omega; \mathbb{R})$. Note that when $u|_\Omega \notin \mathcal{C}^2(\Omega; \mathbb{R})$, the left-hand side of the equation in (1.1) is interpreted as a current of bidegree (n, n). That this makes sense when $u \in \mathrm{psh}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ was established by Bedford–Taylor [3], who established an existence and uniqueness theorem for (1.1) with the above-mentioned data for strongly pseudoconvex domains. Furthermore, [6, Theorem 4.1] is a consequence of Theorem 8.3 in [3].

As hinted at in our discussion in Section 1.2, for proving Theorem 1.6:

- We need an existence theorem for the Dirichlet problem (1.1) in which the datum f is far less well-behaved than what is mentioned right after (1.1) (or in the previous paragraph).
- With the data just mentioned, we also require some information on the modulus of continuity of the solutions to (1.1)

The earliest results to provide information of the above-mentioned type are presented in [16]. The result (which relies strongly on the latter work) that we need for our proofs is the following.

Result 3.2 (paraphrasing [7, Theorem 2] by Charabati)**.** *Let $\Omega \subsetneq \mathbb{C}^n$ be a strongly hyperconvex Lipschitz domain. Let $f \in \mathbb{L}^p(D, \mathsf{m}_{2n})$ for some $p > 1$ and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ be the restriction of a function defined on a neighbourhood of $\partial\Omega$ and of class $\mathcal{C}^{1,1}$. Then, the Dirichlet problem (1.1) has a unique solution u, and u belongs to $\mathcal{C}^{0,s}(\overline{\Omega})$ for each $s \in (0, 1/(nq + 1))$, where $1/p + 1/q = 1$.*

In the above result, in saying that a function defined on an open set $\mathscr{O} \subseteq \mathbb{C}^n$ is of class $\mathcal{C}^{1,1}$ we mean that it is of class $\mathcal{C}^1$ and all it first-order partial derivatives satisfy a Lipschitz condition.

3.2. **Miscellaneous analytical results.** We shall now present several supporting results that we shall need in our proofs. We begin with a technical result. In the proof that follows, if $\mathscr{O}$ is an open set in $\mathbb{C}^m$, then $\mathscr{D}'(\mathscr{O}; \mathbb{C})$ will denote the space of all $\mathbb{C}$-valued distributions on $\mathscr{O}$.

Proposition 3.3. *Let D be a bounded domain in $\mathbb{C}^m$, Ω a bounded domain in $\mathbb{C}^n$, $m < n$, and let $F : D \longrightarrow \Omega$ be a proper holomorphic map. Let $\rho \in \mathcal{C}(\Omega) \cap \mathsf{psh}(\Omega)$ and assume ρ is such that, writing*

$$dd^c\rho = (i/2)\sum\nolimits_{\mu,\nu=1}^{n} b_{\mu\overline{\nu}}\, dw_\mu \wedge d\overline{w}_\nu,$$

$b_{\mu\overline{\nu}} \in \mathbb{L}^\infty(\Omega)$ for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Then:

(a) *In the sense of currents*

$$dd^c(\rho \circ F) = (i/2)\sum\nolimits_{j,k=1}^{m}\left(\sum\nolimits_{\mu,\nu=1}^{n}(b_{\mu\overline{\nu}} \circ F)\,\frac{\partial F_\mu}{\partial z_j}\,\overline{\frac{\partial F_\nu}{\partial z_k}}\right) dz_j \wedge d\overline{z}_k.$$

(b) *Suppose F satisfies the condition in Theorem 1.6. If we write $(dd^c(\rho \circ F))^m = \widetilde{f} d\beta_m$, then $\widetilde{f}$ is a function that is non-negative a.e. and is of class $\mathbb{L}^{p/m}(D, \mathsf{m}_{2m})$, p as in Theorem 1.6.*

Proof. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ be a non-negative cut-off function with $\|\chi\|_1 = 1$ and support in $\mathbb{D}^n$ that satisfies $\chi(w) = \chi(|w_1|, \dots, |w_n|)$ for all $w \in \mathbb{C}^n$. Define $\chi_\varepsilon := \varepsilon^{-2n}\chi(\boldsymbol{\cdot}/\varepsilon)$. For $\varepsilon > 0$, write $\Omega_\varepsilon := \{w \in \Omega : \delta_\Omega(w) < \sqrt{n}\varepsilon\}$. Note that $\rho * \chi_\varepsilon$ is defined on Ω_ε. It is well-known that $\rho * \chi_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon) \cap \mathsf{psh}(\Omega_\varepsilon)$, that

$$\rho * \chi_\varepsilon(w) \searrow \rho(w) \text{ as } \varepsilon \searrow 0 \quad \forall w \in \Omega, \tag{3.1}$$

and, since $\rho \in \mathcal{C}(\Omega)$, this convergence is uniform on compact subsets of Ω; see, for instance, [21, Chapter 2]. Define $D^\varepsilon := F^{-1}(\Omega_\varepsilon)$. Note that $D^\varepsilon \Subset D$, since F is proper, and that $(\rho * \chi_\varepsilon) \circ F$ is well-defined on D^ε. Thus, we shall abbreviate $(\rho * \chi_\varepsilon) \circ (F|_{D^\varepsilon})$ as $(\rho * \chi_\varepsilon) \circ F$ and understand that this is defined on D^ε. Moreover, if we fix $\varepsilon_0 > 0$, then $(\rho * \chi_\varepsilon) \circ F \in \mathscr{D}'(D^{\varepsilon_0}; \mathbb{C})$ for every $\varepsilon \in (0, \varepsilon_0)$. Now, if $\eta \in \mathscr{D}(D^{\varepsilon_0}; \mathbb{C})$, i.e., a test function on D^{ε_0}, then, as $\mathrm{supp}(\eta)$ is a compact subset of D^{ε_0}, by the observation following (3.1), we deduce that

$$(\rho * \chi_\varepsilon) \circ F \longrightarrow (\rho \circ F)|_{D^{\varepsilon_0}} \;\text{ in } \mathscr{D}'(D^{\varepsilon_0}; \mathbb{C}),$$

and the above is true for all $\varepsilon_0 > 0$. This implies, since differentiation is a continuous operator on $\mathscr{D}'(D^{\varepsilon_0}; \mathbb{C})$, that

$$\partial^2_{z_j,\overline{z}_k}((\rho * \chi_\varepsilon) \circ F) \longrightarrow \partial^2_{z_j,\overline{z}_k}\big((\rho \circ F)|_{D^{\varepsilon_0}}\big) \;\text{ in } \mathscr{D}'(D^{\varepsilon_0}; \mathbb{C}),$$

for all $j, k : 1 \leq j, k \leq m$, and the above is true for all $\varepsilon_0 > 0$. Finally, since $\varepsilon_0 > 0$ is arbitrary, we have

$$\left\langle \partial^2_{z_j,\overline{z}_k}((\rho * \chi_\varepsilon) \circ F), \eta \right\rangle \longrightarrow \left\langle \partial^2_{z_j,\overline{z}_k}(\rho \circ F), \eta \right\rangle \;\text{ as } \varepsilon \searrow 0 \tag{3.2}$$

for all $j, k : 1 \leq j, k \leq m$, and for any test function $\eta \in \mathscr{D}(D; \mathbb{C})$.

On the other hand, as $(\rho * \chi_\varepsilon) \circ F \in \mathcal{C}^\infty(D^\varepsilon)$, we have

$$\begin{aligned}
\partial^2_{z_j,\overline{z}_k}((\rho * \chi_\varepsilon) \circ F)(z) &= \left\langle \rho,\, (\partial^2_{z_j,\overline{z}_k}\chi_\varepsilon)(F(z) - \boldsymbol{\cdot}) \right\rangle \\
&= \sum_{\mu,\nu=1}^{n} \left\langle \rho,\, \frac{\partial^2 \chi_\varepsilon}{\partial w_\mu \overline{w}_\nu}(F(z) - \boldsymbol{\cdot})\,\frac{\partial F_\mu}{\partial z_j}(z)\,\overline{\frac{\partial F_\nu}{\partial z_k}}(z) \right\rangle \\
&= \sum_{\mu,\nu=1}^{n} \left\langle \partial^2_{w_\mu,\overline{w}_\nu}\rho,\, \chi_\varepsilon(F(z) - \boldsymbol{\cdot}) \right\rangle \frac{\partial F_\mu}{\partial z_j}(z)\,\overline{\frac{\partial F_\nu}{\partial z_k}}(z) \\
&= \sum_{\mu,\nu=1}^{n} \big(b_{\mu\overline{\nu}} * \chi_\varepsilon\big)(F(z))\,\frac{\partial F_\mu}{\partial z_j}(z)\,\overline{\frac{\partial F_\nu}{\partial z_k}}(z) \quad \forall z \in D^\varepsilon.
\end{aligned} \tag{3.3}$$

The second equality above is due to the chain rule while the last equality follows from the definition of the convolution and from our hypothesis. Note that the above is true for all $\varepsilon > 0$ and for every $j, k : 1 \leq j, k \leq m$. Since $b_{\mu\overline{\nu}} \in \mathbb{L}^\infty(\Omega)$ for each $\mu, \nu : 1 \leq \mu, \nu \leq n$ and as Ω, being bounded, has finite Lebesgue measure, $b_{\mu\overline{\nu}} \in \mathbb{L}^1(\Omega, \mathsf{m}_{2n})$ for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Thus, a diagonal argument gives us a sequence $(l_i)_{i\geq 1} \subset \mathbb{N}$ such that

$$\big(b_{\mu\overline{\nu}} * \chi_{1/l_i}\big)(F(z)) \longrightarrow b_{\mu\overline{\nu}}(F(z)) \ \text{ for a.e. } z \in D \text{ as } i \to \infty,$$

and the above holds true for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Now fix a test function $\eta \in \mathscr{D}(D; \mathbb{C})$. Then, as $(\chi_\varepsilon)_{\varepsilon>0}$ is an approximation of the identity and as $\text{supp}(\eta) \Subset D$, there exists a number $C(\eta, j, k) > 0$ such that

$$\|((b_{\mu\overline{\nu}} * \chi_{1/l_i}) \circ F)\, \partial_{z_j} F\, \overline{\partial_{z_k} F}\, \eta\|_1 \leq C(\eta, j, k) \|b_{\mu\overline{\nu}}\|_\infty \|\eta\|_\infty \quad \forall i \in \mathbb{N}.$$

We can thus apply the Dominated Convergence Theorem, which gives

$$\Big\langle ((b_{\mu\overline{\nu}} * \chi_{1/l_i}) \circ F)\, \partial_{z_j} F\, \overline{\partial_{z_k} F}, \eta \Big\rangle \longrightarrow \langle (b_{\mu\overline{\nu}} \circ F)\, \partial_{z_j} F\, \overline{\partial_{z_k} F}, \eta\rangle \ \text{ as } i \to \infty, \tag{3.4}$$

and which holds true for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. From (3.2), (3.3), and (3.4), part (a) follows.

Let us write

$$dd^c(\rho \circ F) = (i/2) \sum\nolimits_{j,k=1}^{m} a_{j\overline{k}}\, dz_j \wedge d\overline{z}_k,$$

Let $\mathscr{E}_{\mu\overline{\nu}}$ denote the set on which either $b_{\mu\overline{\nu}}$ is undefined or $|b_{\mu\overline{\nu}}|$ equals $+\infty$. Then, by hypothesis, $\mathsf{m}_{2n}(\mathscr{E}_{\mu\overline{\nu}}) = 0$ for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Since F is proper, its critical set has zero Lebesgue measure. It is well-known that, owing to the latter property,

$$\mathsf{m}_{2m}(F^{-1}(\mathscr{E}_{\mu\overline{\nu}})) = 0 \quad \forall \mu, \nu : 1 \leq \mu, \nu \leq n.$$

Thus, the set on which either $b_{\mu\overline{\nu}} \circ F$ is undefined or $|b_{\mu\overline{\nu}} \circ F|$ equals $+\infty$ has zero Lebesgue measure for each $\mu, \nu : 1 \leq \mu, \nu \leq n$. Since each $b_{\mu\overline{\nu}}$ is essentially bounded, $b_{\mu\overline{\nu}} \circ F \in \mathbb{L}^\infty(D)$. Then, by part (a) and by our assumption on F, we have

$$a_{j\overline{k}} \in \mathbb{L}^p(D, \mathsf{m}_{2m}) \subseteq \mathbb{L}^m(D, \mathsf{m}_{2m})$$

for each $j, k : 1 \leq j, k \leq m$. By [3, Proposition 2.7], there exists a constant $C_m \neq 0$ such that $\widetilde{f}$ is given by

$$\widetilde{f} = C_m \sum_{\sigma \in S_m} \text{sign}(\sigma) \prod_{j=1}^{m} a_{j\,\overline{\sigma(j)}}\,, \tag{3.5}$$

where S_m denotes the group of permutations of m objects. By part (a), $\widetilde{f}$ is a homogeneous polynomial of degree m whose indeterminates are $\partial_{z_j} F_\mu\, \overline{\partial_{z_k} F_\nu}$, $1 \leq j, k \leq m, 1 \leq \mu, \nu \leq n$, and whose coefficients are $\mathbb{L}^\infty$-functions. Thus, by Hölder's inequality and our assumption on F, $\widetilde{f} \in \mathbb{L}^{p/m}(D, \mathsf{m}_{2m})$. Finally, since $\rho \circ F \in \mathsf{psh}(D)$, $(dd^c(\rho \circ F))^m$ is a positive (m, m)-current and so, in view of (3.5), $\widetilde{f}$ is non-negative a.e. $\square$

The next two results will be vital to proving Theorems 1.1 and 1.8, respectively. Below, and in subsequent sections, $(\mathfrak{H}_{\mathbb{C}}\varphi)$ will denote the complex Hessian of φ and $\langle \boldsymbol{\cdot}, \boldsymbol{\cdot} \rangle$ the standard Hermitian inner product on $\mathbb{C}^n$.

Proposition 3.4. *Let $\Omega \subsetneq \mathbb{C}^n$ be a B-regular domain, and let $f \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ be a non-negative function. Let $\varphi : \mathcal{U} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth function, where $\mathcal{U}$ is a neighbourhood of $\overline{\Omega}$. Assume that $\varphi \in \mathsf{psh}(\mathcal{U})$ and that there exists a constant $\varepsilon > 0$ such that $\langle v, (\mathfrak{H}_{\mathbb{C}}\varphi)(z) v\rangle \geq \varepsilon \|v\|^2$ for every $z \in \Omega$ and $v \in \mathbb{C}^n$. Let $\omega_{f,\varphi}$ be a modulus of continuity of the unique solution to the Dirichlet problem*

$$\left.\begin{aligned} (dd^c u)^n &= f\beta_n, \quad u \in \mathcal{C}(\overline{\Omega}) \cap \mathsf{psh}(\Omega), \\ u|_{\partial\Omega} &= -2\varphi|_{\partial\Omega}, \end{aligned}\right\} \tag{3.6}$$

for the complex Monge–Ampère equation. Then, there exists a constant $c > 0$ such that

$$k_\Omega(z; v) \geq c\,\sqrt{\varepsilon}\, \frac{\|v\|}{\big(\omega_{f,\varphi}(\delta_\Omega(z))\big)^{1/2}}$$

for every $z \in \Omega$ *and* $v \in \mathbb{C}^n$*.*

Proof. Let $u_{f,\varphi}$ denote the unique solution to the Dirichlet problem (3.6), which is guaranteed by the fact that Ω is B-regular. Define

$$\Phi_1(z) := u_{f,\varphi}(z) + \varphi(z) \quad \forall z \in \overline{\Omega}.$$

By a version of Richberg's regularisation theorem (see [6, Theorem 1.1], for instance), there exists a $\mathcal{C}^\infty$-smooth plurisubharmonic function Φ_2 on Ω such that

$$0 \leq \Phi_2(z) - \Phi_1(z) \leq \omega_{f,\varphi}(\delta_\Omega(z)) \quad \forall z \in \Omega. \tag{3.7}$$

Clearly, Φ_2 extends continuously to $\overline{\Omega}$ (we shall refer to this extension as Φ_2 as well) and

$$\Phi_2(z) = -\varphi(z) \quad \forall z \in \partial\Omega. \tag{3.8}$$

Write $\mathsf{u}(z) := \Phi_2(z) + \varphi(z)$ for each $z \in \overline{\Omega}$. Since Φ_2 is plurisubharmonic on Ω, by our condition on $\mathfrak{H}_{\mathbb{C}}\varphi$,

$$\langle v, (\mathfrak{H}_{\mathbb{C}}\mathsf{u})(z)v\rangle \geq \varepsilon\|v\|^2 \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n. \tag{3.9}$$

Fix $z \in \Omega$. As $\partial\Omega$ is compact, there exists a point $\xi_z \in \partial\Omega$ such that $\delta_\Omega(z) = \|z - \xi_z\|$. It follows from (3.7) that

$$\begin{aligned} |\mathsf{u}(z)| &\leq |\Phi_2(z) - \Phi_1(z)| + |\Phi_1(z) + \varphi(z)| \\ &\leq \omega_{f,\varphi}(\delta_\Omega(z)) + |(\Phi_1(z) + \varphi(z)) - (\Phi_1(\xi_z) + \varphi(\xi_z))|. \end{aligned} \tag{3.10}$$

Since φ is $\mathcal{C}^\infty$-smooth on $\mathcal{U} \supseteq \overline{\Omega}$, hence Lipschitz on $\overline{\Omega}$, there exists a constant $C_1 > 0$ such that

$$|(\Phi_1(z) + \varphi(z)) - (\Phi_1(\xi_z) + \varphi(\xi_z))| \leq C_1\omega_{f,\varphi}(\delta_\Omega(z)).$$

Here, we have used the fact that $\|z - \xi_z\| = \delta_\Omega(z)$ and that $\omega_{f,\varphi}$ is a modulus of continuity of u. Combining the last estimate with (3.10), we get

$$|\mathsf{u}(z)| \leq (1 + C_1)\,\omega_{f,\varphi}(\delta_\Omega(z)).$$

The above holds true for each $z \in \Omega$ as z was chosen arbitrarily and as the choice of C_1 depends only on $\omega_{f,\varphi}$.

By (3.8), we have $\mathsf{u}|_{\partial\Omega} = 0$. Thus, by the maximum principle, u is a smooth negative plurisubharmonic function on Ω. Thus, from the last inequality, (3.9), and Result 4.4, we conclude that

$$k_\Omega(z; v) \geq \left(\frac{1}{(1 + C_1)\alpha}\right)^{1/2} \sqrt{\varepsilon}\,\frac{\|v\|}{\big(\omega_{f,\varphi}(\delta_\Omega(z))\big)^{1/2}} \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n,$$

which is the desired lower bound. □

Result 3.5 (Bharali, [5, Proposition 2.1])**.** *Let* $\varphi : [0, r_0) \longrightarrow [0, +\infty]$ *be a function of class of* $\mathbb{L}^1([0, r_0), \mathsf{m}_1)$ *for some* $r_0 \in (0, 1)$*. Let* $g \in \mathcal{O}(\mathbb{D})$ *and assume that*

$$|g'(re^{i\theta})| \leq \varphi(1 - r) \quad \forall r : 1 - r_0 < r < 1 \text{ and } \forall \theta \in \mathbb{R}.$$

Then, g *extends continuously to* $\partial\mathbb{D}$*.*

4. Geometric preliminaries

In this section, we present a few definitions and a key result that plays a vital role in the proof of Theorem 1.8.

4.1. **Definitions on the geometry of domains.** We begin with a definition that was deferred to a later section in our discussion in Section 1.2.

Definition 4.1. The boundary of a domain $\Omega \subsetneq \mathbb{C}^n$ is called a *Lipschitz manifold* if, for each $p \in \partial\Omega$, there exists a neighbourhood $\mathcal{V}_p$ of p, a unitary map U_p, and a $\mathbb{R}^{2n-1}$-open neighbourhood of the origin that is the domain of a Lipschitz function φ_p such that, denoting the affine map $z \longmapsto \mathsf{U}_p(z-p)$ by U^p and writing $\mathsf{z} = (\mathsf{z}_1, \dots, \mathsf{z}_n) := \mathsf{U}^p(z)$, we have

$$\mathsf{U}^p(\mathcal{V}_p \cap \Omega) = \{(\mathsf{z}', \mathsf{z}_n) \in \mathsf{U}^p(\mathcal{V}_p) : \mathsf{Im}(\mathsf{z}_n) > \varphi_p(\mathsf{z}', \mathsf{Re}(\mathsf{z}_n)),\ (\mathsf{z}', \mathsf{Re}(\mathsf{z}_n)) \in \mathsf{dom}(\varphi_p)\}, \text{ and}$$
$$\mathsf{U}^p(\mathcal{V}_p \cap \partial\Omega) = \{(\mathsf{z}', \mathsf{z}_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \mathsf{Im}(\mathsf{z}_n) = \varphi_p(\mathsf{z}', \mathsf{Re}(\mathsf{z}_n)),\ (\mathsf{z}', \mathsf{Re}(\mathsf{z}_n)) \in \mathsf{dom}(\varphi_p)\}.$$

The above is a familiar class of domains in real analysis, with the difference that the domains encountered classically are domains in $\mathbb{R}^N$. Thus, for domains in $\mathbb{C}^n$, the unitary changes of coordinates mentioned in Definition 4.1 replace the orthogonal changes of coordinates in the classical definition. Such domains have many useful properties and have been studied extensively (see, e.g., [1] and the references therein).

An *open right circular cone with aperture* θ is an open subset of $\mathbb{C}^n$ of the form

$$\{z \in \mathbb{C}^n : \mathsf{Re}[\,\langle z, v\rangle\,] > \cos(\theta/2)\|z\|\} =: \Gamma(\theta, v),$$

where v is some unit vector in $\mathbb{C}^n$, $\theta \in (0, \pi)$ (and $\langle \cdot\,,\,\cdot \rangle$ is as introduced in Section 3).

Now, we present another definition that is needed in proving Theorem 1.6.

Definition 4.2. Let $\Omega \subsetneq \mathbb{C}^n$ be a domain. We say that Ω satisfies an *interior-cone condition with aperture* θ if there exist constants $r_0 > 0$, $\theta \in (0, \pi)$, and a compact subset $K \subset \Omega$ such that, for each $z \in \Omega \setminus K$, there exist a point $\xi_z \in \partial\Omega$ and a unit vector v_z such that

- z lies on the axis of the cone $\xi_z + \Gamma(\theta, v_z)$, and
- $(\xi_z + \Gamma(\theta, v_z)) \cap \mathbb{B}^n(\xi_z, r_0) \subset \Omega$.

We say that Ω satisfies a *uniform interior-cone condition* if there exists a $\theta \in (0, \pi)$ such that Ω satisfies an interior-cone condition with aperture θ.

The property defined above is a part of the hypothesis of the next result, which is a generalisation of the classical Hopf Lemma for plurisubharmonic functions.

Result 4.3 (Mercer, [26, Proposition 1.4])**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ that satisfies an interior-cone condition with aperture θ, $\theta \in (0, \pi)$. Let $\rho : \Omega \longrightarrow [-\infty, 0)$ be a plurisubharmonic function. Then, there exists a constant $c > 0$ such that*

$$\rho(z) \leq -c(\delta_\Omega(z))^\alpha$$

(*where* $\alpha = \pi/\theta$) *for every* $z \in \Omega$.

4.2. **Essential propositions.** We begin this section with a result that, through its role in the proof of Proposition 3.4, is at the core of several of our proofs.

Result 4.4 (special case of [30, Proposition 6])**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain. There exists a constant $\alpha > 0$ (which is a universal constant) such that if $z \in \Omega$ and if there exists a negative plurisubharmonic function u on Ω that is of class $\mathcal{C}^2$ and satisfies*

$$\langle v, (\mathfrak{H}_{\mathbb{C}} u)(w) v\rangle \geq c\|v\|^2 \quad \forall (w, v) \in \Omega \times \mathbb{C}^n,$$

for some $c > 0$, then

$$k_\Omega(z; v) \geq \left(\frac{c}{\alpha}\right)^{1/2} \frac{\|v\|}{|u(z)|^{1/2}} \quad \forall v \in \mathbb{C}^n.$$

Our next result is crucial to the proof of Theorem 1.8. A few remarks are in order here. From [6, Theorem 1.7] (which generalises a result by Sibony in [31] to domains with non-$\mathcal{C}^1$ boundaries), a domain $\Omega \Subset \mathbb{C}^n$ is B-regular if and only if Ω admits a plurisubharmonic peak function at each $p \in \partial\Omega$: i.e., a function $u_p : \overline{\Omega} \longrightarrow (-\infty, 0]$ belonging to $\mathcal{C}(\overline{\Omega}) \cap \mathsf{psh}(\Omega)$ satisfying

$$u_p(z) < 0 \;\; \forall z \in \overline{\Omega} \setminus \{p\} \quad \text{and} \quad u_p(p) = 0.$$

Since a bounded $\mathbb{C}$-strictly convex domain is convex, its B-regularity may seem obvious. But recall that

- a convex domain Ω merely admits a holomorphic **weak** peak function at each $p \in \partial\Omega$ (weak peak functions are not germane to our discussion, so we shall skip the definition).
- the polydisc $\mathbb{D}^n$, although convex, is not B-regular when $n \geq 2$.

We will prove Proposition 4.5 by showing that bounded $\mathbb{C}$-strictly convex domains admit a holomorphic peak function at each boundary point. Since this proposition is vital to the proof of Theorem 1.8, we provide a proof of it here. (A word of caution: $\mathbb{C}$-strictly convex domains are not to be confused with $\mathbb{C}$-convex domains.)

Proposition 4.5. *Bounded $\mathbb{C}$-strictly convex domains are B-regular.*

Proof. Let Ω be a bounded $\mathbb{C}$-strictly convex domain. Let $p \in \partial\Omega$. By definition, there exists a support hyperplane of Ω, say $\mathcal{H}$, containing p such that the $\mathbb{C}$-affine hyperplane

$$\widetilde{\mathcal{H}} := p + \big((\mathcal{H} - p) \cap i(\mathcal{H} - p)\big)$$

satisfies $\widetilde{\mathcal{H}} \cap \overline{\Omega} = \{p\}$. Let us write $H = \widetilde{\mathcal{H}} - p$. Let $v \in H^{\perp}$, where the orthogonal complement is with respect to the standard Hermitian inner product $\langle \cdot, \cdot \rangle$, on $\mathbb{C}^n$, and let $\|v\| = 1$. Let $L := \{p + \zeta v : \zeta \in \mathbb{C}\}$. Let proj_L denote the orthogonal projection onto the $\mathbb{C}$-**affine** line L. To clarify: as $p \in L$, $\mathsf{proj}_L(p) = p$. Next, define

$$\omega := \{\zeta \in \mathbb{C} : p + \zeta v \in \mathsf{proj}_L(\Omega)\},$$

which is a bounded domain in $\mathbb{C}$. Now define $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}$ by $\pi(z) := \langle \mathsf{proj}_L(z) - p, v \rangle$. Let $z \in \overline{\Omega}$ and $\mathsf{proj}_L(z) = p + \zeta v$, for some $\zeta \in \mathbb{C}$. Then, $\zeta \in \overline{\omega}$. Therefore, $\pi(z) = \langle \mathsf{proj}_L(z) - p, v \rangle = \langle \zeta v, v \rangle = \zeta \in \overline{\omega}$. Hence, $\pi(\overline{\Omega}) \subseteq \overline{\omega}$. Clearly, $\pi|_{\Omega} : \Omega \longrightarrow \omega$ is holomorphic. Also, note that, $\pi(p) = 0$ and $\widetilde{\mathcal{H}} = \pi^{-1}\{0\}$. Let $(p_j)_{j \geq 1} \subset \Omega$ be such that $p_j \to p$. Since π is continuous, $\pi(p_j) \to \pi(p) = 0$. Now $\pi(p_j) \in \omega$ but $\pi(p) = 0 \notin \omega$ since $\pi^{-1}\{0\} = \widetilde{\mathcal{H}}$ is disjoint from Ω. Therefore, $0 \in \partial\omega$.

Consider two distinct points $x, y \in \omega$. Then, there exist $\widehat{x}, \widehat{y} \in \Omega$ such that $\pi(\widehat{x}) = x$ and $\pi(\widehat{y}) = y$. Let $\mathscr{H}_{x,y} := \pi^{-1}(\{tx + (1-t)y : t \in [0,1]\})$, which is convex. Since $\Omega \cap \mathscr{H}_{x,y}$ is convex and $\widehat{x}, \widehat{y} \in \Omega \cap \mathscr{H}_{x,y}$, it easily follows that $\{tx + (1-t)y : t \in [0,1]\} \subset \omega$. Since x and y were arbitrarily chosen, ω is convex. Hence, ω is a bounded simply connected domain in $\mathbb{C}$ such that $\partial\omega$ is a Jordan curve and such that $0 \in \partial\omega$. Therefore, by the Riemann Mapping Theorem and Carathéodory's extension theorem, there exists a homeomorphism $\psi : \overline{\omega} \longrightarrow \overline{\mathbb{D}}$ such that $\psi|_{\omega} : \omega \longrightarrow \mathbb{D}$ is a biholomorphism, $\psi(\partial\omega) = \partial\mathbb{D}$, and such that $\psi(0) = 1$.

Let f be a non-vanishing holomorphic peak function for $\mathbb{D}$ at 1. Let $g := f \circ \psi \circ \pi|_{\overline{\Omega}}$. Then, $g|_{\Omega}$ is holomorphic. Let $z \in \overline{\Omega} \setminus \{p\}$. Since $\widetilde{\mathcal{H}} = \pi^{-1}\{0\}$ and $\widetilde{\mathcal{H}} \cap \overline{\Omega} = \{p\}$, $\pi(z) \in \overline{\omega} \setminus \{0\}$. Hence, $\psi(\pi(z)) \neq 1$. Therefore, as f is a holomorphic peak function at 1, $|g(z)| < 1$. Clearly, $g(p) = 1$. So, g is a holomorphic peak function for Ω at p. Define the function $u : \overline{\Omega} \longrightarrow (-\infty, 0]$ by $u(z) := \log|g(z)|$. Clearly, u is continuous, $u|_{\Omega}$ is plurisubharmonic, $u(z) < 0$ for $z \in \overline{\Omega} \setminus \{p\}$, and $u(p) = 0$. Thus, u is a plurisubharmonic peak function for Ω at p. As $p \in \partial\Omega$ was arbitrarily chosen, by the remarks involving [6, Theorem 1.7] preceding this proof, the result follows. $\square$

5. The proof of Theorem 1.1

Fix some $f \in \mathcal{C}(\overline{\Omega};\mathbb{R})$ such that f is non-negative. Since Ω is B-regular, there exists a unique plurisubharmonic solution to the equation (1.1) for the boundary data $-2\varphi|_{\partial\Omega} : \partial\Omega \longrightarrow \mathbb{R}$, where φ is a $\mathcal{C}^\infty$-smooth function defined on some neighbourhood $\mathcal{U}_\varphi$ of $\overline{\Omega}$ such that

- $\varphi \in \mathsf{psh}(\mathcal{U}_\varphi)$, and
- $\langle v, (\mathfrak{H}_{\mathbb{C}}\varphi)(z)v\rangle \geq \varepsilon_\varphi\|v\|^2$, for some $\varepsilon_\varphi > 0$, for every $z \in \Omega$ and $v \in \mathbb{C}^n$.

Let us denote the solution by u_φ (for simplicity of notation, as f is fixed). If possible, let u_φ belong to the class $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$. Write $\omega(r) := r^\alpha$ for $r \in [0,\infty)$. Then, by assumption, there exists a constant $C_\varphi > 0$ such that for all $z_1, z_2 \in \overline{\Omega}$

$$|u_\varphi(z_1) - u_\varphi(z_2)| \leq C_\varphi\,\omega(\|z_1 - z_2\|).$$

By Proposition 3.4, we conclude that there exists a constant $M_\varphi > 0$ such that

$$k_\Omega(z;v) \geq \frac{M_\varphi}{\omega(\delta_\Omega(z))^{1/2}} = \frac{M_\varphi}{(\delta_\Omega(z))^{\alpha/2}} \quad \forall z \in \Omega \text{ and } \forall v \in \mathbb{C}^n : \|v\| = 1. \tag{5.1}$$

By the contractivity of the affine embeddings $\mathbb{D} \ni \zeta \longmapsto z_\nu + \big(\mathbf{r}_\Omega(z_\nu;\mathbf{u}_\nu)\zeta\big)\mathbf{u}_\nu$ for the Kobayashi metric, we have

$$k_\Omega(z_\nu;\mathbf{u}_\nu) \leq \frac{1}{\mathbf{r}_\Omega(z_\nu;\mathbf{u}_\nu)} \quad \forall \nu = 1,2,3,\dots.$$

Combining this inequality with (5.1), it follows that

$$\frac{\mathbf{r}_\Omega(z_\nu;\mathbf{u}_\nu)}{\big(\delta_\Omega(z_\nu)\big)^{\alpha/2}} \leq \frac{1}{M_\varphi} \quad \forall \nu = 1,2,3,\dots,$$

which contradicts (1.3). Thus, we have produced a large class of boundary data with the properties stated in Theorem 1.1 for which the solution to (1.1) is not in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1]$. Hence the result. □

6. The proofs of Theorem 1.6 and Theorem 1.5

To prove the above-mentioned theorem, we will need the following lemma:

Lemma 6.1. *Let $D \subsetneq \mathbb{C}^m$ be a bounded domain such that ∂D is a Lipschitz manifold. Let $p \in \partial D$. Then, in the notation of Definition 4.1, but with ψ_p in place of φ_p, there exist a neighbourhood V of p, $V \Subset \mathcal{V}_p$, and a constant $C > 1$ (both depending on p) such that*

$$\delta_D(z) \leq \mathsf{y}(\mathsf{U}^p(z)) \leq C\delta_D(z) \quad \forall z \in V \cap D,$$

where $\mathsf{y}(\mathsf{z}', \mathsf{z}_m) := \mathsf{Im}(\mathsf{z}_m) - \psi_p(\mathsf{z}, \mathsf{Re}(\mathsf{z}_m))$ *for* $(\mathsf{z}', \mathsf{z}_m) \in \mathsf{U}^p(V \cap D)$.

Proof. The first inequality is obvious. So, we will prove the second inequality. There exists a neighbourhood V of p, $V \Subset \mathcal{V}_p$ such that $\operatorname{diam}(V) < \operatorname{dist}(\overline{V}, \mathbb{C}^m \setminus \mathcal{V}_p)$. Due to the latter inequality, for every $z \in V \cap D$, there exists $x_z \in \mathcal{V}_p \cap \partial D$ such that

$$\delta_D(z) = \delta_{\mathcal{V}_p \cap D}(z) = \|z - x_z\|.$$

Since ψ_p is Lipschitz, y is Lipschitz; let C denote its Lipschitz constant. It follows that for every $z \in V \cap D$

$$\mathsf{y}(\mathsf{U}^p(z)) = \|\mathsf{y}(\mathsf{U}^p(z)) - \mathsf{y}(\mathsf{U}^p(x_z))\| \leq C\|\mathsf{U}^p(z) - \mathsf{U}^p(x_z)\| = C\|z - x_z\| = C\delta_D(z).$$

Hence, the result follows. □

We are now in a position to give a proof of Theorem 1.6.

The proof of Theorem 1.6. Fixing $p \in \partial D$, it suffices to show that F extends continuously to a map $\widetilde{F}_p$ on $(U_p \cap \partial D) \cup D$, for some neighbourhood U_p of p. This is because, if we are able to show this, then the expression

$$\widetilde{F}(z) := \begin{cases} F(z), & \text{if } z \in D, \\ \widetilde{F}_p(z), & \text{if } z \in U_p \cap \partial D, \end{cases}$$

would be well-defined. This is because if $p, q \in \partial D$ are distinct points and $(U_p \cap U_q) \cap \partial D \neq \emptyset$, then for each $\xi \in (U_p \cap U_q) \cap \partial D$, we would have

$$\widetilde{F}_p(\xi) = \lim_{D \ni z \to \xi} \widetilde{F}_p(z) = \lim_{D \ni z \to \xi} F(z) = \lim_{D \ni z \to \xi} \widetilde{F}_q(z) = \widetilde{F}_q(\xi).$$

We will establish the above objective in the following three steps.

Step 1. *A preliminary estimate for* $\|F'(z)v\|$ *for every* $z \in D$, *every* $v \in \mathbb{C}^m$.

Since Ω is strongly hyperconvex Lipschitz, by Result 3.2, there exists a unique plurisubharmonic solution to the Dirichlet problem

$$\begin{aligned} (dd^c u)^n &= 0, \quad u \in \mathcal{C}(\overline{\Omega}) \cap \mathsf{psh}(\Omega), \\ u|_{\partial\Omega} &= -2\|w\|^2, \end{aligned}$$

for the complex Monge–Ampère equation belonging to $\mathcal{C}^{0,s}(\overline{\Omega})$ for some $s \in (0,1)$. Let us denote the solution by $\widetilde{u}$. Therefore, there exists a constant $M_0 > 0$ such that for all $w_1, w_2 \in \overline{\Omega}$

$$|\widetilde{u}(w_1) - \widetilde{u}(w_2)| \leq M_0 \|w_1 - w_2\|^s.$$

By Proposition 3.4, we conclude that there exists a constant $M > 0$ such that

$$k_\Omega(w; v) \geq M \frac{\|v\|}{(\delta_\Omega(w))^{s/2}} \quad \forall w \in \Omega, \quad \forall v \in \mathbb{C}^n. \tag{6.1}$$

By the contractivity of the inclusion map $\mathbb{B}^m(z, \delta_D(z)) \hookrightarrow D$ for the Kobayashi metric, we have

$$k_D(z; v) \leq \frac{\|v\|}{\delta_D(z)} \quad \forall z \in D, \quad \forall v \in \mathbb{C}^m. \tag{6.2}$$

Therefore, combining (6.1) and (6.2), for every $z \in D$, every $v \in \mathbb{C}^m$, we have

$$\begin{aligned} &\frac{M\|F'(z)v\|}{\big(\delta_\Omega(F(z))\big)^{s/2}} \leq k_\Omega(F(z); F'(z)v) \leq k_D(z; v) \leq \frac{\|v\|}{\delta_D(z)} \\ &\implies \|F'(z)v\| \leq \frac{1}{M} \frac{\|v\|}{\delta_D(z)} \big(\delta_\Omega(F(z))\big)^{s/2}. \end{aligned} \tag{6.3}$$

Step 2. *A bound for* $\delta_\Omega(F(z))$ *in* (6.3).

Since Ω is a regular strongly hyperconvex Lipschitz domain, there exists a Lipschitz continuous plurisubharmonic function ρ on a neighbourhood Ω' of $\overline{\Omega}$ such that $\rho < 0$ on Ω, $\rho|_{\partial\Omega} \equiv 0$, and satisfies all other conditions in Definition 2.5-(b). Now, Ω satisfies a uniform interior-cone condition. This is a consequence of $\partial\Omega$ being a Lipschitz manifold. Establishing this involves a standard argument and, so, we shall just briefly outline the steps involved:

- If we fix $q \in \partial\Omega$, then, in the notation of Definition 4.1, to each $\mathsf{w} \in \mathsf{U}^q(\mathcal{V}_q \cap \Omega)$, let us associate the point

  $$\xi_{\mathsf{w}} := \big(\mathsf{w}', \mathsf{Re}(\mathsf{w}_n) + i\varphi_q(\mathsf{w}', \mathsf{Re}(\mathsf{w}_n))\big) \in \mathsf{U}^q(\mathcal{V}_q \cap \partial\Omega),$$

  where w will denote points in $\mathsf{U}^q(\mathcal{V}_q)$ (hence, in a small neighbourhood of 0) and $\mathsf{w} =: (\mathsf{w}', \mathsf{w}_n)$.

- There exists a small neighbourhood W_q of 0, $W_q \Subset \mathsf{U}^q(\mathcal{V}_q)$, and constants $s_q > 0$ and $\theta_q \in (0, \pi)$ such that the truncated cone
$$\big(\xi_{\mathsf{w}} + \Gamma(\theta_q, (0, \dots, 0, i))\big) \cap \mathbb{B}^n(\xi_{\mathsf{w}}, s_q) \subset \mathsf{U}^q(\mathcal{V}_q \cap \Omega)$$
for each $\mathsf{w} \in W_q \cap \mathsf{U}^q(\Omega)$, where θ_q is determined by the Lipschitz constant of φ_q.
- By compactness of $\partial\Omega$, there exist $q_1, \dots, q_N \in \partial\Omega$ such that $\bigcup_{j=1}^{N} (\mathsf{U}^{q_j})^{-1}(W_{q_j}) \supset \partial\Omega$ and, for each $x \in K$, one can produce the (truncated) cones that satisfy the condition stated in Definition 4.2, for
$$K = \Omega \setminus \bigcup_{1 \le j \le N} (\mathsf{U}^{q_j})^{-1}(W_{q_j}),$$
with the parameters $\theta = \min\{\theta_{q_1}, \dots, \theta_{q_N}\}$ and $r_0 = \min\{s_{q_1}, \dots, s_{q_N}\}$.

As Ω satisfies a uniform interior-cone condition, by Result 4.3, there exist constants $c_0 > 0$ and $\alpha > 1$ such that
$$\rho(w) \le -c_0(\delta_\Omega(w))^\alpha \quad \forall w \in \Omega. \tag{6.4}$$
Since ρ is continuous and plurisubharmonic on Ω, $\rho \circ F : D \longrightarrow (-\infty, 0)$ is a continuous plurisubharmonic function. We will show that $\rho \circ F$ extends continuously to $\overline{D}$. Let $p' \in \partial D$ and $(x_\nu)_{\nu \ge 1} \subset D$ be a sequence such that $x_\nu \longrightarrow p'$ as $\nu \to \infty$. Let $(x_{\nu_k})_{k \ge 1}$ be an arbitrary subsequence of $(x_\nu)_{\nu \ge 1}$. Since F is proper and Ω is bounded, there exist a subsequence $(x_{\nu_{k_l}})_{l \ge 1}$ and $q' \in \partial\Omega$ such that $F(x_{\nu_{k_l}}) \longrightarrow q'$ as $l \to \infty$. Therefore, since $\rho(q') = 0$, $\rho \circ F(x_{\nu_{k_l}}) \longrightarrow 0$ as $l \to \infty$. Hence, every subsequence of $(\rho \circ F(x_\nu))_{\nu \ge 1}$ has a subsequence that converges to 0, whence we conclude that $\rho \circ F(x_\nu) \longrightarrow 0$ as $\nu \to \infty$. Thus, the function
$$\widetilde{\rho}(z) := \begin{cases} \rho \circ F(z), & \text{if } z \in D, \\ 0, & \text{if } z \in \partial D. \end{cases}$$
is a continuous function on $\overline{D}$.

Since $\rho \circ F$ is a continuous plurisubharmonic function, $(dd^c(\rho \circ F))^m$ is defined in the sense of currents. Write:
$$(dd^c(\rho \circ F))^m = \widetilde{f}\beta_m.$$
By Proposition 3.3, we see that $\widetilde{f}$ is non-negative a.e. and that $\widetilde{f} \in \mathbb{L}^{p/m}(D, \mathsf{m}_{2m})$. Therefore, since $p/m > 1$ and since D is strongly hyperconvex Lipschitz, by Result 3.2, there exists a unique plurisubharmonic solution to the Dirichlet problem
$$\begin{aligned} &(dd^c u)^m = \widetilde{f}\beta_m, \quad u \in \mathcal{C}(\overline{D}) \cap \mathsf{psh}(D), \\ &u|_{\partial D} = 0, \end{aligned}$$
for the complex Monge–Ampère equation and this belongs to $\mathcal{C}^{0,s_0}(\overline{D})$ for some $s_0 \in (0,1)$. Since, by the above discussion, $\widetilde{\rho}$ is a solution to the above Dirichlet problem, by uniqueness, it follows that $\widetilde{\rho} \in \mathcal{C}^{0,s_0}(\overline{D})$. By the $\mathcal{C}^{0,s_0}$ property, there exists a constant $C_0 > 0$ such that for every $z \in D$ we have
$$\begin{aligned} -\rho(F(z)) = -\widetilde{\rho}(z) = |\widetilde{\rho}(z)| &\le C_0(\delta_D(z))^{s_0} \\ \implies \delta_\Omega(F(z)) &\le C_1(\delta_D(z))^{s_*}, \end{aligned} \tag{6.5}$$
where $C_1 := (C_0/c_0)^{1/\alpha} > 0$, $s_* := s_0/\alpha \in (0,1)$, and the last inequality follows from (6.4).

Step 3. *A Hardy–Littlewood-type argument.*

Combining (6.3), (6.5), and from Lemma 6.1, it follows that there exists a neighbourhood V of p (recall that p is as introduced at the beginning of this proof) and a unitary transformation U^p such that

$$\|F'(z)v\| \leq \frac{C_1^{s/2}}{M}\frac{\|v\|}{(\delta_D(z))^{\widetilde{s}}} \leq M^* \frac{\|v\|}{(\mathsf{y}(\mathsf{U}^p(z)))^{\widetilde{s}}} \quad \forall z \in V \cap D, \quad \forall v \in \mathbb{C}^m, \tag{6.6}$$

where $\widetilde{s} := 1 - (ss_*/2) \in (0,1)$ and $M^* := (C^{\widetilde{s}} C_1^{s/2})/M > 0$. From this stage, the argument will resemble, in part, the proof of [2, Theorem 1.5] — which itself is a Hardy–Littlewood-type argument in a non-smooth setting — so we will be brief. Let us define $\mathcal{D} := \mathsf{U}^p(D)$, $\mathcal{V} := \mathsf{U}^p(V)$, $\mathsf{F} = (\mathsf{F}_1, \mathsf{F}_2, \dots, \mathsf{F}_n) := F \circ (\mathsf{U}^p)^{-1}|_{\mathcal{D}}$, and $\mathsf{z} := \mathsf{U}^p(z)$: a new coordinate system in $\mathbb{C}^m$. From (6.6), by applying the chain rule we have

$$\|\mathsf{F}'(\mathsf{z})v\| \leq \tau(\mathsf{y}(\mathsf{z}))\|v\| \quad \forall \mathsf{z} \in \mathcal{V} \cap \mathcal{D}, \quad \forall v \in \mathbb{C}^m, \tag{6.7}$$

where $\tau(x) := M^*/x^{\widetilde{s}}$ for $x > 0$. There exists a neighbourhood $\mathcal{U} \Subset \mathcal{V}$ of $0 \in \mathbb{C}^m$ and a constant $\delta > 0$ such that

$$\left(\bigcup_{\xi \in \mathcal{U} \cap \partial\mathcal{D}} \overline{\mathbb{B}^m(\xi, \delta)} \right) \cap \mathcal{D} \subset \mathcal{V} \cap \mathcal{D}. \tag{6.8}$$

Let $0 < t < t' < \delta$ and $v_0 := (0, \dots, 0, i) \in \mathbb{C}^m$. Since τ is a non-negative Lebesgue integrable function, by (6.7) and the Fundamental Theorem of Calculus, we can conclude that for every $\xi \in \mathcal{U} \cap \partial\mathcal{D}$ and every $j : 1 \leq j \leq n$, the limit

$$\mathsf{F}_j^\bullet(\xi) := \mathsf{F}_j(\xi + t'v_0) - \lim_{t \to 0^+} \int_t^{t'} i \frac{\partial \mathsf{F}_j}{\partial \mathsf{z}_m}(\xi + xv_0)\, dx \tag{6.9}$$

exists and is independent of t'. Now define $\widetilde{\mathsf{F}}_p = (\widetilde{\mathsf{F}}_{p,1}, \widetilde{\mathsf{F}}_{p,2}, \dots, \widetilde{\mathsf{F}}_{p,n}) : (\mathcal{U} \cap \partial\mathcal{D}) \cup \mathcal{D} \longrightarrow \overline{\Omega}$ by

$$\widetilde{\mathsf{F}}_p(z) := \begin{cases} \mathsf{F}^\bullet(\mathsf{z}) = (\mathsf{F}_1^\bullet(\mathsf{z}), \dots, \mathsf{F}_n^\bullet(\mathsf{z})), & \text{if } \mathsf{z} \in \mathcal{U} \cap \partial\mathcal{D}, \\ \mathsf{F}(\mathsf{z}), & \text{if } \mathsf{z} \in \mathcal{D}. \end{cases}$$

We will show that $\widetilde{\mathsf{F}}_p$ is continuous. In particular, it suffices to show that $\widetilde{\mathsf{F}}_p$ is continuous on $\mathcal{U} \cap \partial\mathcal{D}$.

Let $\varepsilon > 0$ be given. Since τ is Lebesgue integrable, there exists $\kappa \in (0, \delta)$ such that $\int_0^\kappa \tau(x)\, dx < \varepsilon/3$. Hence, from (6.9) and (6.7), it follows that

$$|\mathsf{F}_j^\bullet(\xi) - \mathsf{F}_j(\xi + \kappa v_0)| \leq \int_0^\kappa \tau(\mathsf{y}(\xi + xv_0))\, dx = \int_0^\kappa \tau(x)\, dx < \varepsilon/3 \quad \forall \xi \in \mathcal{U} \cap \partial\mathcal{D}. \tag{6.10}$$

Note that, with our choice of κ, the above estimate is independent of $\xi \in \mathcal{U} \cap \partial\mathcal{D}$. Since $\kappa \in (0, \delta)$, by (6.8), the set $S(\kappa) := \{\xi + \kappa v_0 : \xi \in \mathcal{U} \cap \partial\mathcal{D}\}$ is relatively compact in $\mathcal{D}$. As $\mathsf{F}|_{S(\kappa)}$ is uniformly continuous, it is easy to see — this is the essence of the classical "Hardy–Littlewood trick" — that there exists an $r = r(\varepsilon) > 0$ such that, if $\|\xi_1 - \xi_2\| < r$, $\xi_1, \xi_2 \in \mathcal{U} \cap \partial\mathcal{D}$, then $|\mathsf{F}_j^\bullet(\xi_1) - \mathsf{F}_j^\bullet(\xi_2)| < \varepsilon$ for $j = 1, \dots, n$. We conclude that $\mathsf{F}^\bullet$ is uniformly continuous on $\mathcal{U} \cap \partial\mathcal{D}$.

Now fix $\xi = (\xi', \zeta + i\psi_p(\xi', \zeta)) \in \mathcal{U} \cap \partial\mathcal{D}$, $\zeta \in \mathbb{R}$, where ψ_p is as in Lemma 6.1. Let $(\mathsf{z}_\nu)_{\nu \geq 1} \subset (\mathcal{U} \cap \overline{\mathcal{D}}) \setminus \{\xi\}$ be an arbitrary sequence such that $\mathsf{z}_\nu \longrightarrow \xi$ as $\nu \to \infty$. Let us define

$$\mathsf{Z}_\nu := \begin{cases} \mathsf{z}_\nu, & \text{if } \mathsf{z}_\nu \in \partial\mathcal{D}, \\ \mathsf{z}_\nu, & \text{if } \mathsf{z}_\nu = \big(\xi', \zeta + i(x + \psi_p(\xi', \zeta))\big) \text{ for some } x > 0, \\ \pi(\mathsf{z}_\nu), & \text{otherwise,} \end{cases}$$

where $\pi(\mathsf{z}) = \pi(\mathsf{z}', \mathsf{z}_m) := (\mathsf{z}', \mathsf{Re}\mathsf{z}_m + i\psi_p(\mathsf{z}', \mathsf{Re}\mathsf{z}_m))$. Clearly, π is continuous, and hence, $\mathsf{Z}_\nu \longrightarrow \xi$ as $\nu \to \infty$. Therefore, from (6.10), and from the fact that $\mathsf{F}^\bullet$ is uniformly continuous on $\mathcal{U} \cap \partial\mathcal{D}$, we have

$$\lim_{\nu\to\infty} \widetilde{\mathsf{F}}_p(\mathsf{z}_\nu) = \lim_{\nu\to\infty} \widetilde{\mathsf{F}}_p(\mathsf{Z}_\nu) = \mathsf{F}^\bullet(\xi) = \widetilde{\mathsf{F}}_p(\xi).$$

As $\xi \in \mathcal{U} \cap \partial\mathcal{D}$ was arbitrarily chosen (and as $(\mathsf{z}_\nu)_{\nu\geq 1}$, with the stated properties, was arbitrary) we conclude that $\widetilde{\mathsf{F}}_p$ is continuous. Hence, since U^p is an automorphism of $\mathbb{C}^m$, F extends continuously to $(U_p \cap \partial D) \cup D$, where $U_p := (\mathsf{U}^p)^{-1}(\mathcal{U})$ is a neighbourhood of p. Thus, in view of the discussion at the beginning of this proof, the result follows. □

Next, we provide:

The proof of Theorem 1.5. Since, by definition, D and Ω have $\mathcal{C}^2$-smooth boundaries, ∂D and $\partial\Omega$ are Lipschitz manifolds. Since D and Ω are strongly pseudoconvex, they admit $\mathcal{C}^2$-smooth defining functions that are strongly plurisubharmonic. Thus — see [7, Section 2] — both D and Ω are *regular* strongly hyperconvex Lipschitz domains. Hence, all the conditions in the hypothesis of Theorem 1.6 are satisfied, from which the result follows. □

7. The proofs of Theorem 1.8 and Corollary 1.10

To prove Theorem 1.8, we will need the following result:

Result 7.1 (Mercer, [27])**.** *Let Ω be a bounded convex domain in $\mathbb{C}^n$ and let $\psi : \mathbb{D} \longrightarrow \Omega$ be a complex geodesic. There exists a constant $\beta > 1$ and constants $C_1, C_2 > 0$ such that*

$$C_1(1 - |\zeta|) \leq \delta_\Omega(\psi(\zeta)) \leq C_2(1 - |\zeta|)^{1/\beta} \quad \forall \zeta \in \mathbb{D}.$$

We are now in a position to give

The proof of Theorem 1.8. Since Ω is bounded and $\mathbb{C}$-strictly convex, by Proposition 4.5, Ω is B-regular. Therefore, since ω, as given in the hypothesis, is a modulus of continuity of the canonical function, by Proposition 3.4, we conclude that there exists a constant $c > 0$ such that

$$k_\Omega(z; v) \geq c\frac{\|v\|}{\big(\omega(\delta_\Omega(z))\big)^{1/2}} \quad \forall z \in \Omega.$$

Let $\psi : \mathbb{D} \longrightarrow \Omega$ be a complex geodesic. By the previous inequality, for every $\zeta \in \mathbb{D}$, we have

$$c\frac{\|\psi'(\zeta)\|}{\big(\omega(\delta_\Omega(\psi(\zeta)))\big)^{1/2}} \leq k_\Omega(\psi(\zeta); \psi'(\zeta)) \leq k_\mathbb{D}(\zeta; 1) = \frac{1}{1 - |\zeta|^2} \leq \frac{1}{1 - |\zeta|}.$$

Since ω is monotone increasing, by Result 7.1 we have

$$\omega\big(\delta_\Omega(\psi(\zeta))\big) \leq \omega\left(C_2(1 - |\zeta|)^{1/\beta}\right) \quad \forall \zeta \in \mathbb{D}.$$

Now combining the last two inequalities we get

$$\|\psi'(\zeta)\| \leq \frac{1}{c}\frac{\big(\omega(C_2\left(1 - |\zeta|\right)^{1/\beta})\big)^{1/2}}{1 - |\zeta|} \quad \forall \zeta \in \mathbb{D}. \tag{7.1}$$

Let $s := 1/\beta$. Define a function $\tau : [0, 1) \longrightarrow [0, \infty]$ as follows:

$$\tau(x) := \begin{cases} \sqrt{\omega\left(C_2 x^s\right)}/cx, & \text{if } 0 < x < 1, \\ \infty, & \text{if } x = 0. \end{cases}$$

Write $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$. From (7.1), it follows that

$$|\psi_j'(\zeta)| \leq \tau(1 - |\zeta|) \quad \forall \zeta \in \mathbb{D} \text{ and } \forall j : 1 \leq j \leq n. \tag{7.2}$$

Since $\sqrt{\omega}$ satisfies a Dini condition, $\int_0^{\varepsilon}(\sqrt{\omega(x)}/x)\,dx < \infty$ for every $\varepsilon > 0$. From this, by using a change of variables formula for the Lebesgue integral, it is elementary to see that τ is of class $\mathbb{L}^1([0,1), \mathsf{m}_1)$. Therefore, by Result 3.5 and (7.2), we conclude that for every $j : 1 \leq j \leq n$, ψ_j extends continuously to $\partial\mathbb{D}$. Hence, the result follows. ☐

We now undertake the proof of Corollary 1.10.

The proof of Corollary 1.10. Let $\Omega_1, \dots, \Omega_N$ be bounded strongly convex domains in $\mathbb{C}^n$ such that $\Omega := \bigcap_{j=1}^N \Omega_j$ is non-empty. Clearly, Ω is a bounded $\mathbb{C}$-strictly convex domain in $\mathbb{C}^n$. Let ρ_j be a defining function of class $\mathcal{C}^2$ for Ω_j such that the real Hessian of ρ_j is strictly positive definite at each point in $\mathsf{dom}(\rho_j)$. Since Ω_j is strongly convex, such a defining function always exists; see, for instance [22, Lemma 3.1.4]. Each ρ_j satisfies all the properties in Definition 2.5-(a). Now define the function $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ by

$$\rho(z) := \max_{j:1\leq j\leq N} \rho_j(z).$$

Then, ρ is a Lipschitz plurisubharmonic function. It is elementary to see that ρ satisfies both conditions in Definition 2.5-(a). Hence, Ω is a strongly hyperconvex Lipschitz domain. Therefore, by its definition and by Result 3.2, the canonical function for Ω belongs to $\mathcal{C}^{0,s}(\overline{\Omega})$ for some $s \in (0,1)$. Now define $\omega(x) := x^s$ for $x \in [0,\infty)$. Since $\sqrt{\omega}$ satisfies a Dini condition, the result follows from Theorem 1.8-(b). ☐

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