MA 224 : COMPLEX ANALYSIS
SPRING 2026

A SKETCH OF SOLUTIONS TO PROBLEMS 4 AND 5

Instructor: GAUTAM BHARALI **Exam date: FEBRUARY 18, 2026**

Below are sketches of solutions to a couple of problems in the mid-semster exam that were not too well-handled.

4. Let Ω be a non-empty open set in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Two **closed** paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow \Omega$ are said to be *freely homotopic as closed paths* in Ω if there exists a continuous function $\Gamma : [0,1] \times [0,1] \longrightarrow \Omega$ such that

$$\begin{aligned} \Gamma(\boldsymbol{\cdot}, j) &= \gamma_j \quad \text{for } j = 0, 1, \\ \Gamma(0,t) &= \Gamma(1,t) \quad \forall t \in [0,1]. \end{aligned}$$

Note: We do **not** fix $\gamma_0(0) = \gamma_1(0)$ in the above definition (unlike the notion of homotopy of closed paths stated in class in connection with Cauchy's Theorem)!

Now, suppose γ_0, γ_1 are closed piecewise-$\mathcal{C}^1$ paths in Ω that are freely homotopic as closed paths. Then, how are $\int_{\gamma_0} f(z)\,dz$ and $\int_{\gamma_1} f(z)\,dz$ related?

Note. You may freely use **without proof** the fact that if $a \neq b$ are two points in the same connected component of Ω, then there is a piecewise-$\mathcal{C}^1$ path joining a to b.

Sketch of the solution: Write $a = \gamma_0(0)$. If

$$\Gamma(0,t) = a = \Gamma(1,t) \quad \forall t \in [0,1],$$

then it follows from the homotopy form of Cauchy's Integral Theorem that

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz. \tag{1}$$

We claim that (1) *is true in general.*

So, assume that $\Gamma(0,\boldsymbol{\cdot})$ is **non-constant.** Write $\rho := \Gamma(0,\boldsymbol{\cdot})$. We shall establish the above claim in **three** steps. This needs some further preparation. As $[0,1]$ is compact, $\rho([0,1])$ is compact and ρ is uniformly continuous. Thus,

$$\operatorname{dist}(\rho([0,1]), \mathbb{C}\setminus\Omega) =: r > 0,$$

and there exists $n \in \mathbb{Z}_+$ such that

$$|\rho(s_1) - \rho(s_2)| < r \text{ whenever } |s_1 - s_2| < 2/n \text{ and } s_1, s_2 \in [0,1]. \tag{2}$$

Write $Z_j := \rho(j/n)$, $0 \leq j \leq n$. By (2), $\rho([(j-1)/n, j/n]) \subseteq D(Z_{j-1}, r) \subseteq \Omega$, $1 \leq j \leq n$. The second inclusion is due to the definition of r. By convexity of $D(Z_{j-1}, r)$, the image of the path $[Z_{j-1}, Z_j](\tau) := (1-\tau)Z_{j-1} + \tau Z_j$, $0 \leq \tau \leq 0$, is contained in Ω. Define

$$\sigma := [Z_0, Z_1] * [Z_1, Z_2] * \cdots * [Z_{n-1}, Z_n]$$

with the understanding that each $\langle [Z_{j-1}, Z_j] \rangle$ is linearly reparametrized by $[(j-1)/n, j/n]$, $1 \leq j \leq n$. Finally, write

$$\widetilde{\gamma}_1 := \rho * \gamma_1 * (-\rho)$$

with the understanding that the j-th curve above is linearly reparametrized by $[(j-1)/3,\, j/3]$, $j = 1, 2, 3$.

Step 1. $\widetilde{\gamma}_1 \sim \gamma_0$

Define the map $\widetilde{\Gamma} : [0,1] \times [0,1] \longrightarrow \Omega$ by

$$\widetilde{\Gamma}(s,t) \;:=\; \begin{cases} \rho(3s), & \text{if } 0 \leq s \leq (1-t)/3, \\ \Gamma\Big(\frac{3}{1+2t}s - \frac{1-t}{1+2t}, 1-t\Big) & \text{if } (1-t)/3 \leq s \leq (2+t)/3, \\ \rho(3(1-s)), & \text{if } (2+t)/3 \leq s \leq 1. \end{cases}$$

In words, $\widetilde{\Gamma}(\cdot, t)$ denotes a path that represents travelling along $\Gamma(0,\cdot)$ until $\Gamma(0, 1-t)$ at speed 3, then traversing the closed path $\Gamma(\cdot, 1-t)$ at speed $3/(1+2t)$ and returning to $\gamma_0(0)$ along $-\Gamma(0,\cdot)$ at speed 3. **Check** the continuity of Γ by appealing to the Gluing Lemma and the fact that $\Gamma(0,t) = \Gamma(1,t)$ for all $t \in [0,1]$.

The problem with the path $\widetilde{\gamma}$ is that it **may not** be piecewise-$\mathcal{C}^1$. To rectify this, we define

$$\widehat{\gamma}_1 \;:=\; \sigma * \gamma_1 * (-\sigma)$$

with the understanding that the j-th curve above is linearly reparametrized by $[(j-1)/3,\, j/3]$, $j = 1, 2, 3$.

Step 2. $\widehat{\gamma}_1 \sim \widetilde{\gamma}_1$

Work out the details for the above, by writing down a suitable straight-line homotopy, using the fact that both $\rho([(j-1)/n,\, j/n])$ and $\langle [Z_{j-1}, Z_j] \rangle$ are contained in the convex set $D(Z_{j-1}, r)$.

Step 3. *Completing the solution*

We discussed in class that $\sim$ is an equivalence relation. Thus, from Steps 1 and 2 we have $\widehat{\gamma}_1 \sim \gamma_0$. Thus, from the homotopy form of Cauchy's Integral Theorem:

$$\begin{aligned} \int_{\gamma_0} f(z)\,dz \;&=\; \int_{\widehat{\gamma}_1} f(z)\,dz \\ &= \int_{\sigma} f(z)\,dz + \int_{\gamma_1} f(z)\,dz + \int_{-\sigma} f(z)\,dz \\ &= \int_{\sigma} f(z)\,dz + \int_{\gamma_1} f(z)\,dz - \int_{\sigma} f(z)\,dz \\ &= \int_{\gamma_1} f(z)\,dz, \end{aligned}$$

which establishes the claim written above in italics.

5. Let $\varnothing \neq \Omega \subseteq \mathbb{C}$ be a simply-connected open set and assume that $0 \neq \Omega$. By writng down a function defined in terms of suitable integrals, show that Ω admits a branch of the logarithm.

Solution: Pick and fix a point $z_0 \in \Omega$. As $z_0 \neq 0$, we can pick and fix $a \in \mathbb{C}$ such that $e^a = z_0$. Also, for each $z \in \Omega$, **fix** a piecewise-$\mathcal{C}^1$ path $\gamma_{z_0} : ([0,1], 0, 1) \longrightarrow (\Omega, z_0, z)$. Since Ω is simply-connected, it is connected; thus, when $z \neq z_0$, we are **given** that such a γ_z exists. When $z = z_0$, γ_{z_0} is some closed piecewise-$\mathcal{C}^1$ path whose image contains z_0. Define

$$f(z) \;:=\; a + \int_{\gamma_z} \left(\frac{1}{w}\right) dw, \quad z \in \Omega.$$

We will show that f is a branch of the logarithm.

We have proved in class that as Ω is simply connected,

($*$) f does not depend on the choice of γ_z.

Now, given $z \in \Omega$, let $r = r_z > 0$ such that $D(z,r) \subset \Omega$. Let $w \in D(z,r)^*$. Denote by $[z,w]$ the path $[z,w](t) := (1-t)z + tw$, $0 \leq t \leq 1$. By the convexity of $D(z,r)$ and as $D(z,r) \subset \Omega$, image$(\gamma_z * [z,w]) \subsetneq \Omega$. So, in view of ($*$) and a calculation presented in class, we can deduce:

- For any $w \in D(z,r)$:
$$\frac{f(w)-f(z)}{w-z} = \frac{1}{w-z}\int_{[z,w]} \left(\frac{1}{\zeta}\right) d\zeta.$$
- $f'(z) = 1/z$.

As $0 \notin \Omega$, f' is continuous on Ω. Thus $f \in \mathcal{O}(\Omega)$.

Consider the function $h : \Omega \longrightarrow \mathbb{C}$ defined by $h(z) := \exp(f(z))/z$. As $0 \notin \Omega$ and as $f \in \mathcal{O}(\Omega)$, $h \in \mathcal{O}(\Omega)$ and by the quotient rule and the chain rule

$$h'(z) = \frac{z\exp(f(z))f'(z) - \exp(f(z))}{z^2} = 0 \quad \forall z \in \Omega.$$

As Ω is connected, it follows from the conclusion of Problem 2 that $h \equiv C$ for some constant C. But

$$h(z_0) = \exp(f(z_0))/z_0 = \exp(a)/z_0 = 1,$$

which implies that

$$\exp(f) = \mathsf{id}_\Omega. \tag{3}$$

Now, define $G := f(\Omega)$. By the Open Mapping Theorem, G is an open subset of $\mathbb{C}$. By (3), since id_Ω is injective

$$\exp|_G \text{ is injective.}$$

Then, by (3) again, f is the set-theoretic definition of $\left(\exp|_G\right)^{-1}$. Thus, f is a branch of the logarithm.