MATH 224 : COMPLEX ANALYSIS

SPRING 2026

HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: MARCH 20, 2026**

---

1. Let $\gamma_j : [0,1] \longrightarrow \mathbb{C}$, $j = 1,2$, be closed piecewise-$\mathcal{C}^1$ paths. Let $\alpha \in \mathbb{C}$ be such that

$$|\gamma_2(t) - \gamma_1(t)| \,<\, |\alpha - \gamma_1(t)| \quad \forall t \in [0,1].$$

Argue that $\alpha \notin \langle \gamma_j \rangle$ for $j = 1,2$, and show that $W(\gamma_1;\alpha) = W(\gamma_2;\alpha)$.

2. Let $n \in \mathbb{Z}_+ \setminus \{1\}$ and let $p(z) = z^n + \sum_{0 \leq j \leq n-1} a_j z^j$ be a monic polynomial in $\mathbb{C}[z]$ such that $\sum_{0 \leq j \leq n-1} |a_j| < 1$. Prove that p has n zeros counted according to multiplicity and give some information on where in $\mathbb{C}$ these zeros lie.

Now use the above result and a trick to give a new proof of the Fundamental Theorem of Algebra.

3. Let Ω be a bounded domain in $\mathbb{C}$ and let $f : \overline{\Omega} \longrightarrow \mathbb{C}$ be a continuous non-constant function such that $f|_\Omega$ is holomorphic. Assume that $f(\partial\Omega) \subseteq \{w \in \mathbb{C} : |w| = 1\}$. Can f be non-vanishing?

4. Let Ω be a non-empty open set and let $f : \Omega \longrightarrow \mathbb{C}$ be an injective holomorphic function. Show that $f(\Omega)$ is an open set (note that Ω is not necessarily connected). Is f^{-1} holomorphic on $f(\Omega)$? Please **justify** your answer.

5. Let z_1, z_2, z_3 be distinct points in $\overline{\mathbb{C}}^\infty$ and let w_1, w_2, w_3 be distinct points in $\overline{\mathbb{C}}^\infty$. Show that there exists a unique Möbius transformation T such that $T(z_j) = w_j$, $j = 1,2,3$.

6. Define a *sector* to be a set of the form $\{re^{i\phi} : r > 0 \text{ and } -\theta_0 < \phi < \theta_0\}$, where $\theta_0 \in (0,\pi]$. Let D_1 and D_2 be two open discs in $\mathbb{C}$ such that

$$D_1 \cap D_2 \neq \varnothing \;\text{ and }\; \text{neither } D_1 \subseteq D_2 \text{ nor } D_2 \subset D_1.$$

(*a*) Show that there exists a Möbius transformation that maps $D_1 \cap D_2$ onto a sector.

(*b*) Show that there exists a biholomorphic mapping of $D_1 \cap D_2$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

Tip. You may use, **if** at all needed, freely **without proof** the fact that if $\mathscr{C}$ is a circle in $\overline{\mathbb{C}}^\infty$, then $\overline{\mathbb{C}}^\infty \setminus \mathscr{C}$ has two connected components.