MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: MARCH 27, 2026**

1–2. Problems 8 and 9 from the exercises to III–Section 3 of Conway.

3. Let $r \in (0,1)$. Argue that the principal branch of z^r is biholomorphic on its domain, and compute its range.

4. Let $\prod^+$ denote the upper half-plane, i.e. $\prod^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Suppose $f \in \mathcal{O}(\prod^+)$ and $f(\prod^+) \subset \mathbb{D}$ (here $\mathbb{D}$ denotes the open unit disc with centre $0 \in \mathbb{C}$). How large can $|f'(i)|$ be?

5. Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. Using an auxiliary function similar to those used in the derivation of $\mathrm{Aut}(\mathbb{D})$, show that

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \le |f(z)| \le \frac{|f(0)| + |z|}{1 + |f(0)||z|} \quad \forall z \in \mathbb{D}.$$

6. Let $d(z,w) := \left|(z-w)(1-\overline{w}z)^{-1}\right|$ for all $(z,w) \in \mathbb{D} \times \mathbb{D}$. Show that if a, b, c are distinct points in $\mathbb{D}$, then

$$d(a,c) < d(a,b) + d(b,c)$$

Tip. Please use the fact that we may assume that $a = 0$ and $c \in (0,1)$ — giving full justifications for the latter — before attempting any heavy (and ugly) computations.

Remark. The above shows that the Möbius distance on $\mathbb{D}$ is indeed a metric.

7. Compute $\mathrm{Aut}(\mathbb{C})$.

Hint. Make use of Problem 1 from Homework 9 suitably. We have not yet proved any theorems specifically about essential singularities; look up V–Section 1 in Conway for any such theorem **if** needed.