

MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 12

Instructor: GAUTAM BHARALI

Assigned: APRIL 8, 2026

Some of the problems below are related to material **not yet** covered in class as of April 8; such problems are connected to subjects that will be presented in the final week of classes.

1. Let Ω be a non-empty open set in \mathbb{C} and let $b \in \Omega$. Let $r > 0$ be such that $\overline{D(b, r)} \not\subseteq \Omega$. Let $f \in \mathcal{O}(\Omega)$. Show that for each $k \in \mathbb{Z}_+$,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\partial D(b, r)} \frac{f(w)}{(w - z)^{k+1}} dw$$

for every $z \in D(b, r)^*$.

2. Find a biholomorphic map from $\{z \in \mathbb{D} : \operatorname{Re}(z) > 0\}$ onto \mathbb{D} .

3. Let Ω be a domain in \mathbb{C} and let $\{f_n\}_{n \geq 1} \subset \mathcal{O}(\Omega)$ be a sequence of injective functions that converge in $\mathcal{O}(\Omega)$ to f . Show that f is either injective or a constant function. How does the conclusion change if, instead of a domain, we allow Ω to be an arbitrary non-empty open set?

4. Look up Theorem 2.30 in Chapter III of Conway. Using this theorem, show that if $\Omega \subseteq \mathbb{C}$ is a domain with the property that for any $f \in \mathcal{O}(\Omega)$ that is non-vanishing, there exists a function $g_f \in \mathcal{O}(\Omega)$ such that $(g_f(z))^2 = f(z)$ for all $z \in \Omega$, then for each harmonic function $u : \Omega \rightarrow \mathbb{R}$, there exists a harmonic function $v : \Omega \rightarrow \mathbb{R}$ such that $(u + iv) \in \mathcal{O}(\Omega)$.

5. Let Ω be a non-empty open subset of \mathbb{C} and let $f \in \mathcal{O}(\Omega)$ such that $f(z) \neq 0$ for every $z \in \Omega$. Show that $\log |f|$ is harmonic.

6. Problem 2 from the exercises to IX–Section 1 of Conway.

7. Let $a \in \mathbb{C}$ and let f be a function that is holomorphic on $U \setminus \{a\}$ for some neighbourhood U of a . Assume that f has an essential singularity at a . Show that for each $c \in \mathbb{C}$, $\varepsilon > 0$, and each $\delta > 0$ so small that $\operatorname{ann}(a; 0, \delta) \subseteq U$, there exists $b \in D(c, \varepsilon)$ such that the equation $f(z) = b$ has infinitely many roots in $\operatorname{ann}(a; 0, \delta)$.