MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 1

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In solving the next two problems, recall that if $\{a_n\}_{n\geq 1}$ is a sequence in $\mathbb{R}$, then

$$\begin{aligned}\limsup_{n\to\infty} a_n &:= \sup\{x \in \mathbb{R}\cup\{\pm\infty\} : x = \lim_{k\to\infty} a_{n_k} \text{ for some } \{a_{n_k}\}_{k\geq 1} \subset \{a_n\}_{n\geq 1}\},\\ \liminf_{n\to\infty} a_n &:= \inf\{x \in \mathbb{R}\cup\{\pm\infty\} : x = \lim_{k\to\infty} a_{n_k} \text{ for some } \{a_{n_k}\}_{k\geq 1} \subset \{a_n\}_{n\geq 1}\}.\end{aligned}$$

1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two sequences of positive real numbers. Assume that $\{a_n\}_{n\geq 1}$ converges to a point in $(0, +\infty)$. Show that

$$\limsup_{n\to\infty} a_n b_n = \big(\lim_{n\to\infty} a_n\big)\big(\limsup_{n\to\infty} b_n\big).$$

Does the above hold true if the condition of positivity is dropped?

Tip. Please use the *definition* and/or fundamental facts of the lim sup to prove the above. This results in a cleaner proof that does not require dividng the argument into cases.

2. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Define

$$\begin{aligned}A_k &:= \inf\{a_k, a_{k+1}, a_{k+2}, \dots\},\\ B_k &:= \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}.\end{aligned}$$

Show that

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} A_k, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \lim_{k\to\infty} B_k.$$

3. Let Ω be a connected open subset of $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic on Ω. Suppose $f(\Omega)$ is contained in a line in $\mathbb{C}$ that passes through the origin. Show that f is a constant.

Note. You may use, without proof, the following:

Proposition. *Let Ω be a connected open subset of $\mathbb{R}^2$ and $u : \Omega \longrightarrow \mathbb{R}$ a function for which $\partial_x u$ and $\partial_y u$ exist at each point and are continuous on Ω. If $\partial_x u(x,y) = \partial_y u(x,y) = 0$ for each $(x,y) \in \Omega$, then u is a constant.*

4. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series for which $a_n \neq 0$ for every $n \in \mathbb{N}$. Show that if the limit

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

exists in $[0, +\infty]$, then it equals the radius of convergence of the above power series.

5. Define the extensions of the sine and cosine functions to $\mathbb{C}$ as follows:

$$\cos(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!},$$
$$\sin(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Determine the radii of convergence of the two power series. Next, show that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \text{and} \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

for every $z \in \mathbb{C}$, giving appropriate **justifications.**

6. Fix $b \in \mathbb{C}$. Write $f(z) = \sum_{n=0}^{\infty} a_n(z-b)^n$, where the expression on the right-hand side is a power series with radius of convergence $\rho > 0$. Write $f = u + iv$, $u, v : D(b, \rho) \longrightarrow \mathbb{R}$. Prove that u and v are harmonic on $D(b, \rho)$.

7–8. Problems 15 and 19 from the exercises to III–Secn. 2 of Conway.