MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 2

Instructor: GAUTAM BHARALI **Assigned: JANUARY 16, 2026**

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1. Let $m \in \mathbb{Z}_+ \setminus \{1\}$.

(a) Show that there exists a number $\rho > 0$ such that $(1-z)^{-m}$ equals the sum of a power series of the form $\sum_{n=0}^{\infty} a_{m,n} z^n$ for each $z \in D(0,\rho)$, and that ρ is independent of m, without computing any of the $a_{m,n}$.

(b) Give an expression for each $a_{m,n}$ and justify your answer.

2. Let Ω be an open subset of $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let z_0 be a point in Ω at which $f'(z_0) \neq 0$. Using any relevant result that you know **about $\mathbb{R}^N$-valued maps,** show that there is a neighbourhood $U \subset \Omega$ of z_0 on which f is injective and $V := f(U)$ is an open subset of $\mathbb{C}$. Is the (local) inverse $(f|_U)^{-1}$ holomorphic on V?

3. Let $\Omega \subseteq \mathbb{C}$ be a non-empty open set and let $\gamma : [a,b] \to \Omega$ be a piecewise-$\mathcal{C}^1$ path. Let $f \in \mathcal{C}(\Omega;\mathbb{C})$. Show that $\int_\gamma f\,dz$ is invariant under reparametrization: i.e., if $\widetilde{\gamma}$ is a reparametrization of γ, then

$$\int_\gamma f\,dz = \int_{\widetilde{\gamma}} f\,dz.$$

4. For $\varepsilon > 0$, write $\gamma_\varepsilon(\theta) := \varepsilon e^{i\theta}$, $\theta \in [-\pi,\pi]$. Compute

$$\int_{\gamma_\varepsilon} \left(\frac{1}{z}\right) dz.$$

5. Let Ω be an open subset of $\mathbb{C}$ and let $f \in \mathcal{C}(\Omega;\mathbb{C})$. Suppose there exists a function $F \in \mathcal{O}(\Omega)$ such that $F' = f$. Let $\gamma : [a,b] \to \Omega$ be a piecewise-$\mathcal{C}^1$ path. Show that

$$\int_\gamma f\,dz = F(\gamma(b)) - F(\gamma(a)).$$

6–9. Problems 11, 20, 21, and 24 from the exercises to IV–Secn. 1 of Conway.