MATH 224 : COMPLEX ANALYSIS

SPRING 2026

HOMEWORK 3

Instructor: GAUTAM BHARALI **Assigned: JANUARY 23, 2026**

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1. Let Ω be a non-empty open set in $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Let $z \in \Omega$ and define

$$F_\varepsilon(\theta) := f(z + \varepsilon e^{i\theta}), \quad \theta \in [0, 2\pi],$$

for all $\varepsilon > 0$ sufficiently small that F_ε is defined. Prove that $F_\varepsilon \longrightarrow f(z)$ **uniformly** as $\varepsilon \to 0^+$.

Note. You might not have, in the courses you have taken, studied uniform convergence beyond the context of sequences of functions. Thus, a part of this problem involves formulating what the words "$F_\varepsilon \longrightarrow f(z)$ **uniformly** as $\varepsilon \to 0^+$" must mean.

2. Let Ω be a non-empty open subset of $\mathbb{C}$ and let $f : \Omega \longrightarrow \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \longrightarrow \Omega$ be a piecewise-$\mathcal{C}^1$ path in Ω. Let t_j be finitely many points in $[a, b]$:

$$a = t_0 < t_1 < \cdots < t_n = b,$$

such that $\gamma|_{[t_{j-1}, t_j]} \in \mathcal{C}^1([t_{j-1}, t_j])$, $j = 1, \dots, n$. If M is any non-negative continuous function on Ω, we define

$$\int_\gamma M\, d|z| := \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} M \circ \gamma(s) \big\| \big(\gamma|_{[t_{j-1}, t_j]} \big)'(s) \big\| \, ds.$$

Recall that when $M \equiv 1$, the above integral is just the length of the curve $\mathsf{image}(\gamma)$. Show that

$$\left| \int_\gamma f\, dz \right| \leq \int_\gamma |f|\, d|z|.$$

3–4. Problems 7, parts (a)–(c), and 9, parts (a)–(d) from the exercises to IV–Section 2 of Conway.

Note. The next two problems require a little beyond what was presented in class up to January 23.

5. Recall the complex exponential function, denoted by e^z, $z \in \mathbb{C}$.

(a) Fix $a \in \mathbb{R}$. Show that $e^a e^z = e^{a+z}$ for all $z \in \mathbb{C}$.

(b) Using (a), show that $e^z e^w = e^{z+w}$ for all $z, w \in \mathbb{C}$.

Note. It will help to use something that you know about the **real** exponential.

6. Suppose that f is an entire function and that there exist two real numbers $M > 0$ and $p \geq 1$ such that $|f(z)| \leq M(1 + |z|^p)$ $\forall z \in \mathbb{C}$. Describe, giving a **rigorous** argument, all the entire functions that satisfy this growth estimate.