MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 4

Instructor: GAUTAM BHARALI **Assigned: JANUARY 30, 2026**

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1. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be a domain (i.e., a connected open set) and let $f \in \mathcal{O}(\Omega)$. Suppose f maps Ω into the circle $\{z \in \mathbb{C} : |z| = R\}$ for some $R > 0$. Prove that f is a constant.

Note. We had occasion to see the statement of the Open Mapping Theorem during the lecture on January 28, but **not** its proof. Please solve the above *without* using the Open Mapping Theorem.

2. Given an example of a domain $\Omega \subsetneq \mathbb{C}$ and a function $f \in \mathcal{O}(\Omega)$ with the property that there exists a $b \in \Omega$ and a point $z_0 \in \Omega \setminus D(b, \mathsf{dist}(b, \partial\Omega))$ such that — denoting by

$$\sum_{n=0}^{\infty} a_n (z - b)^n \;=:\; S_f(b, z)$$

the formal power series that equals the power-series development of f around b in some disc centered at b — either $S_f(b, z_0)$ is divergent or $S_f(b, z_0) \neq f(z_0)$.

Note. Here $\mathsf{dist}(b, \partial\Omega)$ is the Euclidean distance between b and $\partial\Omega$.

3. Let $\emptyset \neq \Omega \subseteq \mathbb{C}$ be a domain. Note that $(\mathcal{O}(\Omega), +, \cdot)$ — where $+$ and $\cdot$ are the usual (pointwise) sum and product of functions — is a ring. Show that $(\mathcal{O}(\Omega), +, \cdot)$ is an integral domain.

4. Let Ω_1 and Ω_2 be non-empty open subsets of $\mathbb{C}$. Let $f \in \mathcal{O}(\Omega_1)$, let $H : \Omega_2 \longrightarrow \mathbb{R}$ be harmonic on Ω_2 and suppose $f(\Omega_1) \subset \Omega_2$. Is $H \circ f$ harmonic?

5. Problem 9 from the exercises to IV–Section 3 of Conway.

6. For a function $f : \mathbb{C} \longrightarrow \mathbb{C}$, we say that ***f* has two periods** if there exist two complex numbers w_1 and w_2 such that

- the pair $\{w_1, w_2\}$ is $\mathbb{R}$-linearly-independent; and
- $f(z + w_1) = f(z) = f(z + w_2)$ $\forall z \in \mathbb{C}$.

Describe the set of all entire functions having two periods.

7. Let $\emptyset \neq \Omega \subsetneq \mathbb{C}$ be a domain and suppose $f_1, f_2 : \Omega \longrightarrow \mathbb{C}$ be two branches of the logarithm. Show that there exists an $n \in \mathbb{Z}$ such that

$$f_2(z) = f_1(z) + 2\pi i n \quad \forall z \in \Omega.$$