MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 5

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1. Show that there exists no branch of the logarithm on $\mathbb{C} \setminus \{0\}$.

2. Let Ω be a non-empty open subset of $\mathbb{C}$ and let $g \in \mathcal{O}(\Omega)$. Let $a < b \in \mathbb{R}$ and let $\sigma = \sigma_1 + i\sigma_2 : [a, b] \longrightarrow \mathbb{C}$, where σ_1, σ_2 are real-valued, be a $\mathcal{C}^1$-smooth path with $\langle\sigma\rangle \subsetneq \Omega$. Applying the classical Chain Rule to $\varphi : [a, b] \longrightarrow \mathbb{R}^2$, where

$$\varphi(t) := g_{\mathbb{R}} \circ \sigma(t), \ \ t \in [a, b],$$

prove that $\varphi \in \mathcal{C}^1([a, b])$ and

$$\varphi'(t) = \mathbf{j}\big(g'(\sigma(t))\sigma'(t)\big), \ \ t \in [a, b],$$

where $\mathbf{j}$ is the indentification of $\mathbb{C}$ with $\mathbb{R}^2$ given by $\mathbf{j}(z) = (\mathrm{Re}(z), \mathrm{Im}(z))$ for every $z \in \mathbb{C}$.

3. Let $\gamma_j : [0, T_j] \longrightarrow \mathbb{C}$, $j = 1, 2$, be two closed piecewise-$\mathcal{C}^1$ paths such that $\gamma_1(T_1) = \gamma_2(0)$. Recall the constructions $-\gamma_1$ and $\gamma_1 * \gamma_2$ defined in class. Show that:

- (a) For any point $a \notin \langle\gamma_1\rangle$, $W(-\gamma_1; a) = -W(\gamma_1; a)$.
- (b) For any point $a \notin \langle\gamma_1\rangle \cup \langle\gamma_2\rangle$, $W(\gamma_1 * \gamma_2; a) = W(\gamma_1; a) + W(\gamma_2; a)$.

In the next three problems from Conway's book, replace the word "rectifiable" by the phrase "piecewise-$\mathcal{C}^1$" wherever encountered.

4–6. Problems 5–7 from the exercises to IV–Section 5 of Conway.

7. Let Ω be a non-empty open subset of $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let $\gamma : [0, 1] \longrightarrow \Omega$ be a closed piecewise-$\mathcal{C}^1$ path. What can you say about $\int_\gamma f'(z)\, dz$?

Note. You are **not** given that $W(\gamma; a) = 0$ for every $a \in \mathbb{C} \setminus \Omega$!