MATH 224 : COMPLEX ANALYSIS
SPRING 2026
HOMEWORK 8

Instructor: GAUTAM BHARALI **Assigned: MARCH 9, 2026**

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1. Let $\Omega \subseteq \mathbb{C}$ be a non-empty open set and let $f : \Omega \longrightarrow \mathbb{C}$ be a $\mathbb{C}$-differentiable function. Prove Goursat's Theorem — i.e., show that $f \in \mathcal{O}(\Omega)$ — by completing the following outline:

Step 1. Pick an arbitrary $\Delta := \Delta_{0,1}$ — a closed triangular region contained in Ω. Appeal to the Subdivision Lemma stated in Homework 7 to obtain a nested sequence

$$\Delta^{(0)} \supsetneq \Delta^{(1)} \supsetneq \Delta^{(2)} \supsetneq \cdots,$$

where each $\Delta^{(j)} \in \{\Delta_{j,1}, \dots, \Delta_{j,4^j}\}$ — as per the notation of the Subdivision Lemma — and has the properties stated in the latter lemma. Argue that $\bigcap_{j\in\mathbb{N}} \Delta^{(j)}$ is a singleton $\{z_0\}$.

Step 2. Fix $\varepsilon > 0$. Since f is $\mathbb{C}$-differentiable at z_0, there exists a constant $\delta > 0$ such that

$$\left|(z - z_0)^{-1}\big(f(z) - f(z_0)\big) - f'(z_0)\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

Let $j(\delta) \in \mathbb{Z}_+$ be so large that $\operatorname{diam}(\Delta^{(j(\delta))}) < \delta$. Argue that

$$\left|\oint_{\partial\Delta^{(j(\delta))}} f(z)\,dz\right| \leq \varepsilon \oint_{\partial\Delta^{(j(\delta))}} |z - z_0|\,d|z|$$

using the preceding inequality appropriately.

Step 3. Now, using the properties of $\{\Delta^{(j)}\}_{j\geq 0}$, show that $\oint_{\partial\Delta} f(z)\,dz = 0$. Why does this give the desired conclusion?

2. Let $a \in \mathbb{C}$, $\mathcal{A} := \operatorname{ann}(a; R_1, R_2)$, where $0 \leq R_1 < R_2 \leq +\infty$, and $f \in \mathcal{O}(\mathcal{A})$. For any non-empty compact set $K \subsetneq \mathcal{A}$ and any constants $r_1, r_2 > 0$ such that

$$R_1 < r_1 < r_2 < R_2 \quad \text{and} \quad K \subsetneq \operatorname{ann}(a; r_1, r_2),$$

we have shown in class that

$$f(z) = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \oint_{\partial D(a,r_2)} \frac{f(w)}{(w-a)^{n+1}}\,dw \right\} (z-a)^n + \sum_{n=1}^{\infty} \left\{ \frac{1}{2\pi i} \oint_{\partial D(a,r_1)} \frac{f(w)}{(w-a)^{-(n+1)}}\,dw \right\} \frac{1}{(z-a)^n},$$

where the right-hand side above converges **pointwise** to $f(z)$, and is absolutely convergent, for each $z \in K$. Show that the right-hand side converges **uniformly** to f on K.

Hint. Note that the assertion established in class holds true with $\widetilde{K} := \overline{\operatorname{ann}(a; \widetilde{r}_1, \widetilde{r}_2)}$ replacing K, where $r_1 < \widetilde{r}_1 < \widetilde{r}_2 < r_2$.

3. Let $\overline{\mathbb{C}}^{\infty}$ denote the one-point compactification of $\mathbb{C}$ (note that $\mathbb{C}$ is locally compact and Hausdorff). For each point $(a_1, a_2, a_3) \in S^3 \setminus \{(0,0,1)\}$, where S^3 is the unit sphere in $\mathbb{R}^3$ with centre $(0,0,0)$, let

$$\big(X(a_1,a_2,a_3), Y(a_1,a_2,a_3), 0\big) := \text{the intersection of the unique line in } \mathbb{R}^3 \text{ through } (0,0,1) \text{ and } (a_1,a_2,a_3) \text{ with the plane } \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 0\}.$$

Define the map $f : S^3 \longrightarrow \overline{\mathbb{C}}^{\infty}$ by

$$f(a_1, a_2, a_3) = \begin{cases} X(a_1, a_2, a_3) + iY(a_1, a_2, a_3), & \text{if } (a_1, a_2, a_3) \in S^3 \setminus \{(0,0,1)\}, \\ \infty, & \text{if } (a_1, a_2, a_3) = (0,0,1). \end{cases}$$

Prove that f is a homeomorphism between S^3 and $\overline{\mathbb{C}}^{\infty}$.

4. Let $\Omega \subseteq \mathbb{C}$ be a non-empty open set and let $S \subsetneq \Omega$ be a non-empty set that has no limit points in Ω. Let $f : (\Omega \setminus S) \longrightarrow \mathbb{C}$ be a holomorphic function that has a pole at each $a \in S$. Show that f extends to a continuous function $\widetilde{f} : \Omega \longrightarrow \overline{\mathbb{C}}^{\infty}$.

5–8. Problems 4, 5, 13, and 16 from the exercises to V–Section 1 of Conway.