MA328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES

AUTUMN 2024

HOMEWORK 1

Instructor: GAUTAM BHARALI **DUE: Tuesday, Sep. 10, 2024**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems **and briefly state the nature of the help obtained.**

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$\begin{aligned} |\alpha| &:= \alpha_1 + \dots + \alpha_n \quad \text{and} \quad \alpha! := \alpha_1! \dots \alpha_n!, \\ z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}, \\ \frac{\partial^{|\alpha|}}{\partial z^\alpha} &:= \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}. \end{aligned}$$

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1. Let $m \in \mathbb{Z}_+$. For each $z \in \mathbb{C}^m$, write $z_j = x_j + iy_j,\ x_j, y_j \in \mathbb{R}$, and denote this correspondence between $\mathbb{C}^m$ and $\mathbb{R}^{2m}$ as $\mathbb{C}^m \leftrightarrow \mathbb{R}^{2m}$. Let $\mathcal{B}^m := (\epsilon_1, \dots, \epsilon_m)$ be the standard ordered basis on $\mathbb{C}^m$, and let $\mathcal{B}^{2m}(\mathbb{R}) = (e_1, e_2, \dots, e_{2m})$ be the $\mathbb{R}$-basis of $\mathbb{R}^{2m}$ with the properties:

$$\begin{aligned} \epsilon_j &\leftrightarrow e_{2j-1}, \\ i\epsilon_j &\leftrightarrow e_{2j}, \quad j = 1, \dots m. \end{aligned}$$

Fix $n \in \mathbb{N} \setminus \{0, 1\}$. Let $T : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^2$ be a **real** linear transformation and let

$$[T]_{\mathcal{B}^{2n}(\mathbb{R}),\, \mathcal{B}^2(\mathbb{R})} = \begin{bmatrix} a_{11} & b_{11} & a_{12} & b_{12} & \dots a_{1n} & b_{1n} \\ a_{21} & b_{21} & a_{22} & b_{22} & \dots a_{2n} & b_{2n} \end{bmatrix}$$

be the matrix representation of T with respect to the $\mathbb{R}$-bases defined above. Derive a necessary and sufficient condition involving the numbers $\{a_{ij}, b_{ij} : i = 1, 2,\ 1 \leq j \leq n\}$ for T to be a $\mathbb{C}$-linear functional.

2. Consider the power series

$$\sum_{\alpha \in \mathbb{N}^n} C_\alpha z^\alpha.$$

Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ be such that $\xi_j \neq 0$, $j = 1, \dots, n$, and suppose the above power series converges at ξ (**note:** absolute convergence is *not* assumed). Then, show that this power series converges absolutely at each point, and uniformly on any compact subset, of the polydisc $\Delta(\xi)$, where $\Delta(\xi) := D(0, |\xi_1|) \times \dots \times D(0, |\xi_n|)$.

3. Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, be an open set and let $f, g : \Omega \longrightarrow \mathbb{C}$ be $\mathbb{C}$-differentiable. Show that fg is $\mathbb{C}$-differentiable. Now, deduce an expression for $\partial(fg)/\partial z_j$, $j = 1, \dots, n$, in terms of the $\partial/\partial z_i$ derivatives of f and g, where $i = 1, \dots, n$.

4. Let $X_1, \dots X_n \in \mathbb{C}$ and suppose $|X_j| < 1$ for $j = 1, \dots, n$. Show that $\sum_{\alpha \in \mathbb{N}^n} X^\alpha$ is absolutely convergent and that

$$\sum_{\alpha \in \mathbb{N}^n} X^\alpha = \prod_{j=1}^{n} \Big(\sum_{\nu \in \mathbb{N}} X_j^\nu \Big).$$

Hint. Although this is **not** the **only** approach to the solution, consider defining an auxiliary function that you know is holomorphic on $\mathbb{D}^n$, $\mathbb{D}$ being the open unit disc with centre $0 \in \mathbb{C}$.

5. Let Ω be an open subset of $\mathbb{C}^n$ and let $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let $a \in \Omega$. Let $(r_1, \dots, r_n) \in (\mathbb{R}_+)^n$ be such that $\overline{D(a_1, r_1) \times \dots \times D(a_n, r_n)} \subset \Omega$. Argue that

$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial D(a_1, r_1)} \dots \int_{\partial D(a_n, r_n)} \frac{f(\zeta)}{\prod_{j=1}^n (\zeta_j - a_j)^{\alpha_j + 1}} \, d\zeta_n \dots d\zeta_1.$$

6. Prove Liouville's theorem in $\mathbb{C}^n$, $n \geq 2$, **without** using Cauchy's estimates.

7. Prove the following version of Leibniz's Theorem:

Theorem. *Let M be a smooth, compact, connected, oriented manifold and let $\phi : M \times \Omega \longrightarrow \mathbb{C}$, where Ω is a domain in $\mathbb{R}^m$. Let m denote a "Lebesgue measure" on M (i.e., **fix** a volume form, which exists as M is oriented, and use this as a gauge to get $m(S)$ for any Borel set $S \subseteq M$). Suppose:*

(a) $\phi \in \mathcal{C}(M \times \Omega)$, and

(b) $\partial\phi/\partial y_j$ exists for $j = 1, \dots, m$ and

$$\frac{\partial \phi}{\partial y_j} \in \mathcal{C}(M \times \Omega) \ \ \textit{for } j = 1, \dots, m.$$

Then, the function

$$\psi(y) := \int_M \phi(x, y) \, dm(x), \ \ y \in \Omega,$$

is of class $\mathcal{C}^1(\Omega)$. Furthermore,

$$\frac{\partial \psi}{\partial y_j}(y) = \int_M \frac{\partial \phi}{\partial y_j}(x, y) \, dm(x) \ \ \forall y \in \Omega \textit{ and } j = 1, \dots, m.$$

8. Let $\Omega \subsetneq \mathbb{C}$ be a planar domain. Show that there exists a function $f \in \mathcal{O}(\Omega)$ that does not admit a domain $G \supsetneq \Omega$ or a function $F_f \in \mathcal{O}(G)$ such that $F_f|_\Omega = f$.

Hint. Construct a sequence $\{a_n\} \subset \Omega$ that has no limit points in Ω such that

$$\overline{\{a_n : n = 1, 2, 3, \dots\}} \setminus \{a_n : n = 1, 2, 3, \dots\} = \partial\Omega.$$

How can the above — or a result in 1-variable complex analysis — be used to construct the desired f?

In the next two problems, $\mathbb{D}$ will denote the open unit disc with centre $0 \in \mathbb{C}$.

9. Let Ω be a connected, open neighbourhood of the set $\mathbf{H} := \overline{\mathbb{D}} \times \{0_{\mathbb{C}^{n-1}}\} \cup \partial\mathbb{D} \times \mathbb{D}^{n-1}$, $n \geq 2$. Define:

$$\mathscr{C} := \text{the set of connected components of } (\overline{\mathbb{D}} \times \mathbb{D}^{n-1}) \cap \Omega \text{ that do } \textit{not} \text{ contain } \mathbf{H},$$

$$\omega := \left[\text{the connected component of } (\overline{\mathbb{D}} \times \mathbb{D}^{n-1}) \cap \Omega \text{ that contains } \mathbf{H}\right]^\circ,$$

$$\widetilde{\Omega} := \mathbb{D}^n \bigcup \left(\Omega \setminus \left(\overline{\textstyle\bigcup_{V \in \mathscr{C}} V}\right)\right).$$

Show that for each $f \in \mathcal{O}(\Omega)$, there exists a function $F_f \in \mathcal{O}(\widetilde{\Omega})$ such that $F_f|_\omega = f|_\omega$.

10. Given two domains Ω_1 and Ω_2 in $\mathbb{C}^n$, a holomorphic map $F : \Omega_1 \longrightarrow \Omega_2$ is called a *biholomorphism* if F is bijective and F^{-1} is also holomorphic. Given a domain Ω in $\mathbb{C}^n$, an *automorphism of* Ω is a biholomorphism of Ω onto itself. Now, describe all the automorphisms of $(\mathbb{D} \times \mathbb{C})$.