MA328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES
AUTUMN 2024
HOMEWORK 2

Instructor: GAUTAM BHARALI **DUE: Thursday, Oct. 31, 2024**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems **and briefly state the nature of the help obtained.**

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$
\begin{aligned}
|\alpha| &:= \alpha_1 + \cdots + \alpha_n\,, \\
\alpha! &:= \alpha_1! \dots \alpha_n!\,, \\
z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}\,.
\end{aligned}
$$

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1. Show that the $\overline{\partial}$-problem

$$\frac{\partial u}{\partial \overline{z}} = \phi,$$

where $\phi \in \mathcal{C}^1_{\mathsf{c}}(\mathbb{C})$ does **not** necessarily have a compactly-supported solution.
Hint. First consider the solution

$$u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(w)}{w - z} dA(w),$$

for an appropriately chosen ϕ.

2. Consider the function $f(z, w) := \frac{1}{1-(z+w)}$.

a) Find the power-series development of f, call it $\mathcal{S}_f$, in some small neighbourhood of $(0, 0) \in \mathbb{C}^2$.

b) Find the domain of convergence of the series (i.e., $\mathscr{D}(\mathcal{S}_f)$ in the notation used in class) that you computed in part (*a*).

3. Let $n \geq 2$, $0 < r_1, r_2 < 1$, and write

$$\Omega := \big(D(0, r_1) \times \mathbb{D}^{n-1}\big) \cup \big(\mathbb{D}^{n-1} \times D(0, r_2)\big).$$

Describe **explicitly**, in terms of r_1 and r_2, a Reinhardt domain $\widetilde{\Omega} \supsetneq \Omega$ such that for each $f \in \mathcal{O}(\Omega)$, there exists $F_f \in \mathcal{O}(\widetilde{\Omega})$ such that $F_f|_\Omega = f$.

4. Let Ω_1 and Ω_2 be domains in $\mathbb{C}^n$ and let $F : \Omega_1 \to \Omega_2$ be a biholomorphism of Ω_1 onto Ω_2. Show that if Ω_1 is a domain of holomorphy, then so is Ω_2.

5. Let $\Omega_j \subseteq \mathbb{C}^{n_j}$, $j = 1, 2$, be domains of holomorphy. Show that the open set $\Omega_1 \times \Omega_2$ is a domain of holomorphy.

6. Let Ω be a non-empty open subset of $\mathbb{C}$. Let $\{u_\alpha\}_{\alpha\in A}$ denote a non-empty family of subharmonic functions on Ω. If the function

$$U(z) := \sup_{\alpha\in A} u_\alpha(z) \quad \forall z \in \Omega$$

is upper-semicontinuous, then show that $U \in \mathsf{sh}(\Omega)$.

7. Let S be a non-empty subset of $\mathbb{R}^N$ such that no point of S is an isolated point, and let u be an upper semicontinuous function on S. Let K be a compact subset of S. Show that there exists a point $a \in K$ such that $\sup_{x\in K} u(x) = u(a)$.

Note. The above statement presupposes the fact, discussed in class, that u is bounded above on compact sets.

8. Let Ω be a non-empty open subset of $\mathbb{C}$ and let $u \in \mathsf{sh}(\Omega)$. Suppose $\kappa : \mathbb{R} \to \mathbb{R}$ is an increasing convex function. Define

$$U(z) := \begin{cases} \lim_{t\to-\infty} \kappa(t), & \text{if } u(z) = -\infty, \\ \kappa \circ u(z), & \text{otherwise.} \end{cases}$$

Show that $U \in \mathsf{sh}(\Omega)$.

Hint. You need to consider a suitable theorem in measure theorem involving the integral and convex functions.

9. Let Ω be a non-empty open subset of $\mathbb{C}^n$ and let $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a plurisubharmonic function. Let $\varphi : \mathbb{D} \to \Omega$ be a holomorphic map. Show that $U \circ \varphi \in \mathsf{sh}(\mathbb{D})$.

10. Let Ω be a non-empty open subset of $\mathbb{C}^n$ on which the Kontinuitätssatz holds true. Show that for any $\mathbb{C}$-quasinorm μ and any map $\varphi \in \mathcal{C}(\overline{\mathbb{D}}; \Omega)$ such that $\varphi|_{\mathbb{D}}$ is holomorphic,

$$\mu(\varphi(\overline{\mathbb{D}}); \Omega^{\mathsf{c}}) = \mu(\varphi(\partial\mathbb{D}); \Omega^{\mathsf{c}}).$$