MA 328 : INTRODUCTION TO SEVERAL COMPLEX VARIABLES

AUTUMN 2024

HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Tuesday, Nov. 19, 2024**

Note:

a) You are allowed to discuss these problems with your classmates, but individually-written and **original** write-ups are expected for submission. Please **acknowledge** any persons from whom you received help in solving these problems **and briefly state the nature of the help obtained.**

b) Given a multi-index $\alpha \in \mathbb{N}^n$, we shall use the following notation:

$$\begin{aligned} |\alpha| &:= \alpha_1 + \cdots + \alpha_n\,, \\ \alpha! &:= \alpha_1! \dots \alpha_n!\,, \\ z^\alpha &:= z_1^{\alpha_1} \dots z_n^{\alpha_n}\,. \end{aligned}$$

c) The notation $I \in \mathscr{I}_q$ means that I is an increasing q-tuple in $\{1, \dots, n\}^q$.

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1. Let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{C}^n$, $n \geq 2$. Let $\rho : (0, +\infty) \to (0, +\infty)$ be an increasing, concave function of class $\mathcal{C}^2$. Show that the function

$$u(z) := \begin{cases} -\rho\big(-\log(\|z\|)\big), & \text{if } z \in \mathbb{B}^n \setminus \{0\}, \\ -\big(\lim_{x\to+\infty} \rho(x)\big), & \text{if } z = 0, \end{cases}$$

is a plurisubharmonic function on $\mathbb{B}^n$. Please be clear that your argument **actually** conforms to the definition!

2. Let Ω be a non-empty open subset of $\mathbb{C}^n$ and let U be plurisubharmonic on Ω. Prove that for each $a \in \Omega$, $U(a) = \limsup_{z\to a} U(z)$.

3. Let X be an n-dimensional complex manifold and let $\mathfrak{A} = \{(U_\alpha, \psi_\alpha) : \alpha \in A\}$ be the complex structure on X. Let TX be the classical (i.e., real) tangent bundle of X obtained by viewing $\mathfrak{A}$ as a $\mathcal{C}^\infty$-smooth atlas. Recall that this means that — denoting by π the bundle-projection of TX onto X — we have:

- for each $\alpha \in A$, homeomorphisms h_α such that the diagrams

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{R}^{2n} \arrow[d, "\mathsf{proj}_1"] \\ & U_\alpha \end{tikzcd}$$

  commute (here proj_1 denotes the projection onto the first factor); and

- smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(2n, \mathbb{R})$, for all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, that determine the transition functions for TX;

that are canonically determined by the collection $\{\psi_\alpha : \alpha \in A\}$. Write

$$T^{\mathbb{C}}X := \bigcup\nolimits_{x\in X}(T_xX)\otimes\mathbb{C}.$$

Show, using $\mathfrak{A}$, that $T^{\mathbb{C}}X$ can be endowed with the structure of a smooth complex vector bundle whose fibres are $2n$-dimensional complex vector spaces.

4. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and let $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a densely-defined unbounded (linear) operator. Show that T is a closed operator if and only if $\mathsf{Dom}(T)$ is a Hilbert space when equipped with the graph norm.

5. Let $q \geq 1$. Recall that the *formal adjoint* of $\overline{\partial} : \mathbb{L}^2_{(0,q-1)}(\Omega;\phi_1) \longrightarrow \mathbb{L}^2_{(0,q)}(\Omega;\phi_2)$, (where $\overline{\partial}$ is defined in the sense of distributions) is the adjoint of $\overline{\partial}$ on smooth $(0,q)$-forms "paired against $(\mathcal{C}^\infty_{\mathsf{c}})^{0,q-1}(\Omega)$." Given $\alpha \in \mathscr{I}_{q-1}$ and $\beta \in \mathscr{I}_q$, define

$$\varepsilon^{j\beta}_{\alpha} := \begin{cases} 0, & \text{if } j \in \alpha, \\ 0, & \text{if } \{j\}\cup\alpha \neq \beta, \\ & \\ \operatorname{sgn}\begin{pmatrix} j & \alpha_1 & \dots & \alpha_{q-1} \\ \beta_1 & \beta_2 & \dots & \beta_q \end{pmatrix}, & \text{if } \{j\}\cup\alpha = \beta, \end{cases}$$

where $1 \leq j \leq n$. Denoting by $\overline{\partial}^*$ formal adjoint of $\overline{\partial} : \mathbb{L}^2_{(0,q-1)}(\Omega;\phi_1) \longrightarrow \mathbb{L}^2_{(0,q)}(\Omega;\phi_2)$, show that

$$\overline{\partial}^*_{q-1}\left(\sum_{\beta\in\mathscr{I}_q} f_\beta d\overline{z}^\beta\right) = \sum_{\alpha\in\mathscr{I}_{q-1}}\left\{\sum_{\beta\in\mathscr{I}_q}\sum_{j=1}^n \varepsilon^{j\beta}_{\alpha} e^{\phi_1-\phi_2}\left(f_\beta\frac{\partial\phi_2}{\partial z_j} - \frac{\partial f_\beta}{\partial z_j}\right)\right\} d\overline{z}^\alpha,$$

where $\sum_{\beta\in\mathscr{I}_q} f_\beta d\overline{z}^\beta \in (\mathcal{C}^\infty)^{0,q}(\Omega)$.