MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS

AUTUMN 2025

HOMEWORK 1

Instructor: GAUTAM BHARALI **DUE: Tuesday, September 16, 2025**

Remarks and instructions :

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

c) You are allowed to use AI tools but: (*i*) You must **acknowledge** the use of AI tools for each problem for which such tools were used — citing the name of the tool used and the manner in which it was used, and (*ii*) Be aware that LLMs perform poorly on queries involving unusual formulations of problems or problems using terminology that mean different things in different mathematical disciplines.

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1. Let X be a non-empty set and $f : X \to X$ a function. Define $F : X \times \mathbb{N} \to X$ by

$$F(x, n) := \begin{cases} x, & \text{if } n = 0, \\ \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x), & \text{if } n \in \mathbb{Z}_+, \end{cases}$$

for every $x \in X$. Show that the dynamical structure $(F, \mathbb{N})$ on X satisfies the semigroup property.

2. Let X be a Hausdorff, 2-dimensional, locally Euclidean space. Suppose $\mathscr{A} := \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ and $\mathscr{A}^* := \{(V_\beta, \psi_\beta) : \beta \in J\}$ be two holomorphic atlases on X. Show that $\mathscr{A} \sim \mathscr{A}^*$ if and only if id_X is a biholomorphic map $\mathsf{id}_X : (X, \mathscr{A}) \to (X, \mathscr{A}^*)$ between Riemann surfaces (**to elaborate:** this means that we use the atlases $\mathscr{A}$ and $\mathscr{A}^*$ to encode the holomorphicity of the latter map, taking $\mathscr{A}$ as the atlas on the domain and $\mathscr{A}^*$ as the atlas on the range).

3. Let X be a Riemann surface with a holomorphic atlas $\mathscr{A} := \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$. Let $f : X \to \mathbb{C}$ be a holomorphic function. Show that the chart-wise conditions for holomorphicity of f remain true with

$$(\psi_\beta, V_\beta) \;\text{ replacing }\; (\phi_\alpha, U_\alpha)$$

in these chart-wise conditions if $\mathscr{A}^* := \{(V_\beta, \psi_\beta) : \beta \in J\}$ is another holomorphic atlas on X with $\mathscr{A} \sim \mathscr{A}^*$.

4. Let X and Y be Riemann surfaces and let $f : X \to Y$ be a holomorphic map. Formulate a statement of the Open Mapping Theorem in this setting and prove it.

5. The 1-dimensional *complex projective space*, denoted by $\mathbb{P}^1$, is defined as

$$\mathbb{C}^2 \setminus \{(0,0)\} / \sim,$$
$$\text{where } (x_0, x_1) \sim (y_0, y_1) \iff (y_0, y_1) = \lambda(x_0, x_1) \text{ for some } \lambda \in \mathbb{C} \setminus \{0\},$$

equipped with the quotient topology. Let $[x_0 : x_1]$ denote the equivalence class of (x_0, x_1) and let $U_j := \{[x_0 : x_1] \in \mathbb{P}^1 : x_j \neq 0\}$, $j = 0, 1$.

a) Write $\phi_0 : U_0 \ni [x_0 : x_1] \longmapsto x_1/x_0$. Show that the expression for ϕ_0 does not depend on the choice of representative of $[x_0 : x_1]$, and that ϕ_0 is a homeomorphism.

b) With part (*a*) as a guide, construct a holomorphic atlas $\mathscr{A} := \{(U_0, \phi_0), (U_1, \phi_1)\}$ on $\mathbb{P}^1$.

c) Prove that the Riemann surfaces $(\mathbb{P}^1, \mathscr{A})$ and $\overline{\mathbb{C}}^{\infty}$ (i.e., the Riemann sphere) are biholomorphic.

6. Let ω_1 and ω_2 be two non-zero complex numbers that are $\mathbb{R}$-independent when viewed as vectors in $\mathbb{R}^2$. Let us view the (real) 2-dimensional torus $\mathbb{T}^2$ as $\mathbb{T}^2 = S^1 \times S^1$ (equipped with the relative topology that it inherits from $\mathbb{C} \times \mathbb{C}$). Let us write

$$U_{00} := \{(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2 : 0 < \theta_1, \theta_2 < 2\pi\},$$
$$\phi_{00} : U_{00} \ni (e^{i\theta_1}, e^{i\theta_2}) \longmapsto \frac{\theta_1}{2\pi}\omega_1 + \frac{\theta_2}{2\pi}\omega_2.$$

You may **assume without proof** that U_{00} is open in $\mathbb{T}^2$ and that ϕ_{00} is a homeomorphism onto its image in $\mathbb{C}$. Emulating the above definition, and the fact that the same point in $\mathbb{T}^2 \hookrightarrow \mathbb{C}^2$ can be represented by

$$(e^{i(\theta_1+2\pi\mu)}, e^{i(\theta_2+2\pi\nu)}) \text{ for some } \theta_1, \theta_2 \in [0, 2\pi) \text{ and } \forall(\mu, \nu) \in \mathbb{Z}^2,$$

construct **four** charts (one of which is given above)

$$(U_{jk}, \phi_{jk}), \quad \phi_{jk} : U_{jk} \to \mathbb{C}, \quad j = 0, 1, \; k = 0, 1,$$

that cover $\mathbb{T}^2$ such that

$$\mathscr{A} := \{(U_{ij}, \phi_{jk}) : j = 0, 1, \; k = 0, 1\}$$

is a holomorphic atlas on $\mathbb{T}^2$.

7. Explicitly describe all the elements of $\mathrm{Hol}(\overline{\mathbb{C}}^{\infty}; \overline{\mathbb{C}}^{\infty})$.
Hint. If $F \in \mathrm{Hol}(\overline{\mathbb{C}}^{\infty}; \overline{\mathbb{C}}^{\infty})$, then consider the meromorphic function $F|_{\mathbb{C}}$ and explore what singularities the latter can or cannot have at ∞.

8. Let $p : Y \to X$ be a covering space, where X is a Riemann surface. Fix a holomorphic atlas $\mathscr{A} := \{(U_\alpha, \phi_\alpha) : \alpha \in J\}$ on X. For each $x \in X$, **fix** an evenly covered neighbourhood V_x of x and write $\mathscr{I} := \{(\alpha, x) \in J \times X : U_\alpha \cap V_x \neq \varnothing\}$. Show that

$$\mathscr{B} := \bigcup_{x \in X} \bigcup_{y \in p^{-1}\{x\}} \left\{ \left(p^{-1}(U_\alpha \cap V_x)\big|^y, \phi_\alpha \circ \left(p|_{p^{-1}(U_\alpha \cap V_x)|^y} \right) \right) : (\alpha, x) \in \mathscr{I} \right\}$$

is a holomorphic atlas on Y, where we define

$$p^{-1}(U_\alpha \cap V_x)\big|^y := \text{the connected component of } p^{-1}(U_\alpha \cap V_x) \text{ containing } y.$$

9. Let $(X, \mathscr{A}_X)$ and $(Y, \mathscr{A}_Y)$ be connected Riemann surfaces and let $p : Y \to X$ be a holomorphic covering map. Let $(Z, \mathscr{A}_Z)$ be a connected Riemann surface and let $f : Z \to X$ be a holomorphic map. Suppose there exists a lift $\widetilde{f} : Z \to Y$ of f. Show that $\widetilde{f}$ is holomorphic.

10. Let $\mathscr{M}_{\mathbb{D}}$ denote the Möbius function on $\mathbb{D} \times \mathbb{D}$. Show that if a, b, c are distinct points in $\mathbb{D}$, then

$$\mathscr{M}_{\mathbb{D}}(a, c) \;<\; \mathscr{M}_{\mathbb{D}}(a, b) + \mathscr{M}_{\mathbb{D}}(b, c)$$

Tip. Please use the fact that we may assume that $a = 0$ and $c \in (0, 1)$ — giving full justifications for the latter — before attempting any heavy computations.