MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS
AUTUMN 2025
HOMEWORK 2

Instructor: GAUTAM BHARALI **DUE: Thursday, October 23, 2025**

Remarks and instructions:

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

c) You are allowed to use AI tools but: (*i*) You must **acknowledge** the use of AI tools for each problem for which such tools were used — citing the name of the tool used and the manner in which it was used, and (*ii*) Be aware that LLMs perform poorly on queries involving unusual formulations of problems or problems using terminology that mean different things in different mathematical disciplines.

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1. Let $\mathsf{p}_{\mathbb{D}}$ denote the Poincaré distance. Show that for any holomorphic map $f : \mathbb{D} \to \mathbb{D}$,

$$\mathsf{p}_{\mathbb{D}}(f(z_1), f(z_2)) \leq \mathsf{p}_{\mathbb{D}}(z_1, z_2) \quad \forall z_1, z_2 \in \mathbb{D}.$$

Note. Please do not use the formula for $\mathsf{p}_{\mathbb{D}}$ to prove the above; the derivation of the formula **uses** the above inequality!

2. Let (X, d) be a metric space. Show that if (X, d) has the Heine–Borel property, then it is Cauchy-complete.

3. Prove that the Riemann sphere $\overline{\mathbb{C}}^{\infty}$ is the unique (up to biholomorphic equivalence) Riemann surface uniformized by $\overline{\mathbb{C}}^{\infty}$.

Tip. If you are unable to prove this without the use of **just** the results presented in class, then you may: (a) assume without proof that Riemann surfaces are oriented surfaces, and (b) appeal to the topological classification of compact orientable surfaces.

4. Let (X, d) be a metric space that has the Heine–Borel property. Let $f : X \to X$ be a map that satisfies

$$d(f(x), f(y)) < d(x, y) \quad \forall x \neq y, \ x, y \in X.$$

Suppose f has a unique fixed point x_0. Show that $f^n \longrightarrow x_0$ uniformly on compact subsets of X.

5. Let X be a compact hyperbolic Riemann surface. Provide details to the following sketch to show that $\mathrm{Aut}(X)$ is finite.

(a) Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ be a holomorphic atlas on X. You are given the following result:

If X is compact and hyperbolic, $\mathrm{Aut}(X)$ *has the structure of a compact* ***complex*** *Lie group. Moreover, with respect to the differentiable structure on X determined by $\mathscr{A}$ by treating ϕ_α as a $\mathbb{R}^2$-valued map for each I and with respect to the analogous differentiable structure on* $\mathrm{Aut}(X)$*, the map* $\Phi : \mathrm{Aut}(X) \times X \to X$ *given by*

$$\Phi(g, x) := g(x) \quad \forall (g, x) \in \mathrm{Aut}(X) \times X$$

is $\mathcal{C}^\infty$-smooth.

Reasearch what it means for a function $f : \Omega \to \mathbb{C}$, Ω an non-empty open subset of $\mathbb{C}^n$, $n \in \mathbb{Z}_+$, to be *holomorphic*; deduce from this and **just state** (i) what it means for a Hausdorff, $2n$-dimensional locally-Euclidean space Y to be an n-dimensional complex manifold, (ii) what it means for a map $F : Y \to X$ to be holomorphic; and use the above result to show that Φ is holomorphic.

(b) Formulate a several-variables version of a result that we know for holomorphic functions in one variable that is relevant to the present problem and, combining it with the conclusion of part (a), deduce that $\mathrm{Aut}(X)$ is finite.

6. Give a **rigorous** and complete proof that any element $f \in \mathrm{Aut}(\mathbb{C})$ is of the form $f(z) = az + b$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$.

7. Let $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ be a non-constant holomorphic map, $f \not\equiv \mathsf{id}_{\overline{\mathbb{C}}^\infty}$, having a non-empty Fatou set. Let Ω be a connected component of the Fatou set. Show that $f(\Omega)$ is also a connected component of the Fatou set of f.

8. Let $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ be a rational map and let ξ be a fixed point of f. Show the following relationship between the relevant multipliers:

$$\lambda_f(\xi) = \lambda_{\tau \circ f \circ \tau^{-1}}\big(\tau(\xi)\big),$$

for any Möbius transformation τ.

9. Let $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ be a rational map, f non-constant and $f \neq \mathsf{id}_{\overline{\mathbb{C}}^\infty}$. Let z_0 be a periodic point with period $p \in \mathbb{Z}_+ \setminus \{1\}$. Show that each of the periodic points $z_1, \ldots, z_{p-1}$ has the same classification as z_0.