MATH 380 : INTRODUCTION TO COMPLEX DYNAMICS
AUTUMN 2025
HOMEWORK 3

Instructor: GAUTAM BHARALI **DUE: Tuesday, November 11, 2025**

Remarks and instructions:

a) You are allowed to discuss these problems with your fellow-students, but individually-written and **original** write-ups are expected for submission.

b) Please **acknowledge** any persons from whom you received help in solving these problems — stating the problem(s) in which you took their help.

c) You are allowed to use AI tools but: (*i*) You must **acknowledge** the use of AI tools for each problem for which such tools were used — citing the name of the tool used and the manner in which it was used, and (*ii*) Be aware that LLMs perform poorly on queries involving unusual formulations of problems or problems using terminology that mean different things in different mathematical disciplines.

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1. This problem draws upon:

- the notation and construction in Problem 6 of the first asignment;
- the conclusion of Problem 2 of the first assignment (although there **are** other ways to solve the problem).

Write $\boldsymbol{\omega} := (\omega_1, \omega_2)$, where ω_1 and ω_2 are two non-zero complex numbers that are $\mathbb{R}$-independent when viewed as vectors in $\mathbb{R}^2$. Let

$$\mathscr{A}^{\boldsymbol{\omega}} := \left\{(U^{\boldsymbol{\omega}}_{ij}, \phi^{\boldsymbol{\omega}}_{jk}) : j = 0, 1,\ k = 0, 1\right\}$$

denote the holomorphic atlas on the torus $\mathbb{T}^2 = S^1 \times S^1$ as introduced in Problem 6 of the first assignment. Now write $\boldsymbol{\tau} := (\tau_1, \tau_2)$, where τ_1 and τ_2 are also two non-zero complex numbers that are $\mathbb{R}$-independent. Find a necessary condition on $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ such that $\mathscr{A}^{\boldsymbol{\omega}} \sim \mathscr{A}^{\boldsymbol{\tau}}$. **Do note:** you can use without proof — but you have to be **correct** — your conclusions from Problem 6 of the first asignment.

Remark: Unless you have stated an overly permissive condition, the condition that you have discovered is also sufficient — but this can be a bit laborious to show.

2. Study Section 4.3 from the book by Beardon (and Appendix II in case you require further information on the Weierstrass $\wp$-function) for an example of a rational map $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ whose Julia set is $\overline{\mathbb{C}}^\infty$.

Remark: Notice the connection between the material in Appendix II and Problem 1 above.

3. Let $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ be a rational map having a repelling periodic orbit. Show that this orbit lies in the Julia set of f.

4. Let $f : \overline{\mathbb{C}}^\infty \to \overline{\mathbb{C}}^\infty$ be a rational map and assume that $\deg(f) \geq 2$.

(a) Let $z_0 \in \overline{\mathbb{C}}^{\infty}$ have finite grand orbit under f. Using the following result (the **local** version of which you have studied as the "counting zeros theorem"):

> For a point $w \in \overline{\mathbb{C}}^{\infty}$, $\mathsf{Card}\big(f^{-1}\{w\}\big) < \deg(f)$ if and only if at least one $z \in f^{-1}\{w\}$ is a critical point of f.

show that each point in $GO_f(z_0)$ is a critical point of f.

(b) Show that $\mathscr{E}_f$ is a union of superattracting periodic orbits.

Hint. You may use without proof, if required, the fact that $\deg(f^n) = \deg(f)^n$.

5. Let $f : \overline{\mathbb{C}}^{\infty} \to \overline{\mathbb{C}}^{\infty}$ be a rational map that fixes 0 and assume that 0 is a repelling fixed point. Show that there exists a holomorphic map $\varphi : \mathbb{C} \to \overline{\mathbb{C}}^{\infty}$, satisfying $\varphi(0) = 0$, that is biholomorphic on a small neighbourhood of zero and such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \mathsf{mult}_\lambda\ } & \mathbb{C} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \overline{\mathbb{C}}^{\infty} & \xrightarrow[f]{\quad} & \overline{\mathbb{C}}^{\infty} \end{array}$$

commutes. Here, $\mathsf{mult}_\lambda : z \longmapsto \lambda z$, where $\lambda = f'(0)$.

6. Let $f \in {}_0\mathcal{O}$ and suppose f has a zero of multiplicity $N \geq 2$ at $z = 0$. Assume that

$$f(z) = z^N(1 + b(z)) \text{ in a neighbourhood of } 0$$

where $b \in {}_0\mathcal{O}$ such that $b = 0$. Let $r_1 > 0$ be so small that $|b(z)| < 1/2$ and $|f(z)| \leq (3/4)|z|$ for all $z \in D(0, r_1)$. Write

$$\mathbb{H} := \{w \in \mathbb{C} : \mathsf{Re}(w) < \log(r_1)\},$$

and recall, from the proof of Böttcher's normalization that there exists a holomorphic map $\Lambda : \mathbb{H} \to \mathbb{H}$ such that the diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\quad \Lambda \quad} & \mathbb{H} \\ \exp\big\downarrow & & \big\downarrow\exp \\ D(0, r_1)^* & \xrightarrow[f]{\quad} & D(0, r_1)^* \end{array}$$

commutes. Write $\Phi_n := \Lambda^n / N^n$, $n \in \mathbb{N}$. Show that the sequence $\{\Phi_n\}_{n \geq 0}$ is uniformly Cauchy on compact subsets of $\mathbb{H}$.