UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 10

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 4, 2021**

---

1. Let $a < b$ be real numbers and let $f \in \mathcal{R}\big([a,b]\big)$. Let $c_1, c_2, c_3 \in [a,b]$ — not necessarily distinct or in ascending order. Then show that

$$\int_{c_1}^{c_3} f(x)dx \;=\; \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx.$$

Note. By a problem in Homework 9, we know that all the integrals above exist.

2. You are given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous and satisfies

$$\int_0^x f(t)dt \;=\; 1 + x^2 + x\sin(2x) \quad \forall x \in \mathbb{R}.$$

Compute $f(\pi/4)$.

3–4. Solve Problems 17 and 22 from Section 5.5 of Apostol.

5. Let $g : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $g(x) = 1/x$, $x \neq 0$. Define the function $L(x) = \log|x|$ $\forall x \neq 0$.

a) Argue **rigorously** that $L|_{(-\infty,0)}$ is a primitive of the function $g|_{(-\infty,0)}$.

b) Based on our discussion on the Leibnizian notation and the meaning of the left-hand side below, **justify** the equation:

$$\int \frac{1}{x}\,dx \;=\; \log|x| + C.$$

6. Let $x > 0$ and $\alpha \in \mathbb{Q}$. Recall that we have previously given the definition of x^α in class. Prove that $x^\alpha = e^{\alpha\log(x)}$.

7. This problem is meant to demonstrate the diversity of forms in which vector spaces arise. Let $V = (0, \infty)$, let $\oplus$ denote the sum of two elements in V, and let $\odot$ denote the scalar multiplication, where the scalar field is $\mathbb{R}$, according to the following definition:

$$x \oplus y \;=\; xy \;\; \text{(the usual multiplication in } \mathbb{R}) \quad \forall x, y \in V,$$
$$c \odot x \;=\; x^c \quad \forall c \in \mathbb{R}, \text{ and } \forall x \in V.$$

Prove that V is a vector space over the scalar field $\mathbb{R}$ with the zero vector being 1.

Hint. Although this is a problem in linear algebra, you will need to use something from an earlier topic!

Since there is **only one tutorial ahead in the semester** it would be good to have some problems from linear algebra for discussion in that tutorial. Thus, following problems will go a little beyond what has been taught until now, and anticipate parts of the lectures of **February 5** and **February 8**.

8. Solve Problems 14, 16, 18, and 20 from Section 15.9 of Apostol, omitting **for the moment** the computation of dimensions.

9. By following the reference to the proof of Theorem 12.8 in the discussion of the following result in Apostol, give a proof of the following:

Theorem (THEOREM 15.5 of Apostol)**.** *Let V be a vector space over the field $\mathbb{F}$ and let $\varnothing \neq S \subset V$. Let S have n elements, $n \in \mathbb{N} - \{0\}$. Then any finite subset of $L(S)$ with more than n elements is linearly dependent.*