UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 10 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Let $a < b$ be real numbers and let $f \in \mathcal{R}\big([a,b]\big)$. Let $c_1, c_2, c_3 \in [a,b]$ — not necessarily distinct or in ascending order. Then show that

$$\int_{c_1}^{c_3} f(x)dx \;=\; \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx.$$

Note. By a problem in Homework 9, we know that all the integrals above exist.

Sketch of solution: The solution to this problem has many cases that need to be discussed separately.

Case 1: $c_1 = c_2 = c_3$.
Since, by definition we have

$$\int_A^A f(x)\,dx = 0$$

for any $A \in [a,b]$, the given equation is trivially true in this case.

Case 2: $c_1 = c_3$, $c_2 \neq c_1 (= c_3)$.
In this case, by the convention we have adopted,

$$\int_{c_1}^{c_2} f(x)dx = -\int_{c_2}^{c_3} f(x)dx,$$

while the left-hand side equals 0. The desired equation follows.

Case 3: $c_1 \neq c_3$, $c_2 = c_1$ or c_3.
In this case, one of the integrals on the right-hand side equals 0 while the other equals $\int_{c_1}^{c_3} f(x)dx$. The desired equation follows.

The above leaves us with exactly one more case to consider, namely:

Case 4: c_1, c_2 and c_3 are distinct.
This case involves several sub-cases. Before we address them, let us write

$$A_1 := \min(c_1, c_2, c_3), \quad A_3 := \max(c_1, c_2, c_3),$$

and label the lone element of $\{c_1, c_2, c_3\} - \{A_1, A_3\}$ as A_2. By addivity with respect to interval of integration, we have

$$\int_{A_1}^{A_3} f(x)dx \;=\; \int_{A_1}^{A_2} f(x)dx + \int_{A_2}^{A_3} f(x)dx. \tag{1}$$

We can now tackle all sub-cases in 2-3 lines each.

For example, consider the case $c_3 < c_1 < c_2$. Then, by (1), we have

$$\begin{aligned} \int_{c_3}^{c_2} f(x)dx &= \int_{c_3}^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx \\ \Rightarrow -\int_{c_2}^{c_3} f(x)dx &= -\int_{c_1}^{c_3} f(x)dx + \int_{c_1}^{c_2} f(x)dx \\ \Rightarrow \int_{c_1}^{c_3} f(x)dx &= \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{c_3} f(x)dx. \end{aligned}$$

2. You are given a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous and satisfies

$$\int_0^x f(t)dt = x^2 + x\sin(2x) \quad \forall x \in \mathbb{R}.$$

Compute $f(\pi/4)$.

Solution: Let us pick real numbers $a < b$ such that $0, \pi/4 \in (a, b)$. As $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f|_{[a,b]} \in \mathcal{R}([a, b])$. We can now apply the 1st Fundamental Theorem of Calculus to $f|_{[a,b]}$ to get

$$\left[\int_0^{(\cdot)} f(t)dt\right]'(x) = f(x) \quad \forall\, x \in (a, b).$$

Thus, for any $x \in (a, b)$

$$f(x) = 2x + \sin 2x + 2x\cos 2x \quad [\text{by the sum and product rules}].$$

In particular, since $\pi/4 \in (a, b)$, $f(\pi/4) = (\pi/2) + 1$.

3–4. Solve Problems 17 and 22 from Section 5.5 of Apostol.

Sketch of solution: Parts $(a), (b)$ and (d) of Problem 22 are solved using the 1st Fundamental Theorem of Calculus, and arguing exactly as above, with the difference that we apply this theorem to

$$f|_{[0,b]}\,,$$

where b is some constant, $b > 2$, and $x \in (0, b)$. Part (c) is elementary; figure it out yourself.

We now tackle Problem 17. In this case, we pick and fix $x \in \mathbb{R}$ arbitrarily. Now pick $a < b$, two real numbers, such that $0, 1, x \in (a, b)$. As f is continuous on $\mathbb{R}$, $f|_{[a,b]} \in \mathcal{R}([a, b])$, and $(\cdot)^2 f|_{[a,b]} \in \mathcal{R}([a, b])$. We can thus apply the 1st Fundamental Theorem of Calculus to get

$$\begin{aligned} &\left[\int_0^{(\cdot)} f(t)dt\right]'(x) = f(x), \\ &\left[\int_{(\cdot)}^1 t^2 f(t)dt\right]'(x) = \left[-\int_1^{(\cdot)} t^2 f(t)dt\right]'(x) = -x^2 f(x). \end{aligned}$$

Since $x \in \mathbb{R}$ was arbitrary, the above is true $\forall\, x \in \mathbb{R}$. So, differentiating both sides of the given equation gives:

$$\begin{aligned} f(x) &= -x^2 f(x) + 2x^{15} + 2x^{17} \quad \forall\, x \in \mathbb{R} \\ \Rightarrow (1 + x^2)f(x) &= 2x^{15}(1 + x^2) \\ \Rightarrow f(x) &= 2x^{15} \quad \forall\, x \in \mathbb{R}. \end{aligned}$$

Finally, by substituting $x = 0$ into the given equation, we have

$$\int_0^1 2t^{17} dt + c = 0$$
$$\Rightarrow c = -1/9.$$

5. Let $g : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $g(x) = 1/x$, $x \neq 0$. Define the function $L(x) = \log|x|$ $\forall x \neq 0$.

a) Argue **rigorously** that $L|_{(-\infty,0)}$ is a primitive of the function $g|_{(-\infty,0)}$.

b) Based on our discussion on the Leibnizian notation and the meaning of the left-hand side below, **justify** the equation:
$$\int \frac{1}{x}\, dx \;=\; \log|x| + C.$$

Sketch of solution: When $x \in (-\infty, 0)$, $\log|x|$ is well-defined and so

$$L(x) = \log(-x) \quad \forall\, x \in (-\infty, 0).$$

As log is differentiable at each point in $x \in (0, +\infty)$ and the function $x \longmapsto -x$ is differentiable at each point in $(-\infty, 0)$, the Chain Rule tells us that $L'(x)$ exists at $x \in (-\infty, 0)$ and

$$L'(x) = \frac{1}{-x}(-1) = \frac{1}{x} = g(x) \quad \forall\, x \in (-\infty, 0).$$

Thus, L satisfies the definition of primitive of $g|_{(-\infty,0)}$. This establishes (a).

Now, by the meaning of Leibnizian notation and the fact, proved in class, that $(\log)'(x) = 1/x\ \forall\, x \in (0, \infty)$, we get

$$\int \frac{1}{x}\, dx = \log x + c$$
$$\Rightarrow \int \frac{1}{x}\, dx = \log|x| + c,\ x \in (0, \infty). \tag{2}$$

By the meaning of Leibnizian notation and part (a), we also get

$$\int \frac{1}{x}\, dx = \log|x| + c,\ x \in (-\infty, 0). \tag{3}$$

From (2) and (3), we have the desired result.

6. Let $x > 0$ and $\alpha \in \mathbb{Q}$. Recall that we have previously given the definition of x^α in class. Prove that $x^\alpha = e^{\alpha \log(x)}$.

Sketch of solution: Since the exponential is the inverse function of log, we have

$$x^\alpha = e^{\log(x^\alpha)} \quad (x > 0). \tag{4}$$

We can apply the function log to x^α since, by definition, $x^\alpha > 0$. If we can show that $\log(x^\alpha) = \alpha \log(x)$, then, by (4):
$$x^\alpha = e^{\log(x^\alpha)} = x^\alpha = e^{\alpha \log(x)}, \text{ as required.}$$

Thus, we have the following **goal:** to show that $\log(x^\alpha) = \alpha \log(x)$. To this end, fix:

$$p \in \mathbb{Z} \quad \text{and} \quad q \in \mathbb{N} - \{0\}$$

such that $\alpha = p/q$. we will prove the following claims:

Claim 1. $\log(y^p) = p \log(y)\ \forall y \in (0, +\infty)$.
If $p \in \mathbb{N}$, then show that this claim follows by mathematical induction. Now consider $p \in \mathbb{Z} - \mathbb{N}$. We then have

$$\begin{aligned} 0 = \log(1) &= \log(y^p \cdot y^{-p}) \\ &= \log(y^p) + \log(y^{-p}) \\ &= \log(y^p) - \log(y^p) \quad \forall\, y > 0 \qquad\qquad [\text{since } -p \in \mathbb{N}\,] \\ \Rightarrow \log(y^p) &= p \log(y) \quad \forall\, y > 0 \end{aligned}$$

when $p \in \mathbb{Z} - \mathbb{N}$ as well. This establishes Claim 1.

Claim 2. $\log(y^{1/q}) = q^{-1} \log(y) \quad \forall\, y \in (0, +\infty)$.
By definition $y^{1/q}$ denotes that unique $a > 0$ such that $a^q = y$. By Claim 1,

$$\begin{aligned} q \log(a) &= \log(a^q) = \log(y) \\ \Rightarrow q \log(y^{1/q}) &= \log(y) \qquad\qquad [\text{by the meaning of } y^{1/q}\,] \\ \Rightarrow \log(y^{1/q}) &= q^{-1} \log(y) \end{aligned}$$

and this is true for any $y > 0$. Hence Claim 2.

To establish our goal, recall that $x^\alpha := (x^p)^{1/q}$. Thus, taking $y = x^p$ in Claim 2, we get

$$\log(x^\alpha) = q^{-1} \log(x^p).$$

Now, application of Claim 1 gives

$$\log(x^\alpha) = (pq^{-1}) \log(x) = \alpha \log(x).$$

The stated **goal** is thus established, and thus the result.

7. This problem is meant to demonstrate the diversity of forms in which vector spaces arise. Let $V = (0, \infty)$, let $\oplus$ denote the sum of two elements in V, and let $\odot$ denote the scalar multiplication, where the scalar field is $\mathbb{R}$, according to the following definition:

$$\begin{aligned} x \oplus y &= xy \ \ (\text{the usual multiplication in } \mathbb{R}) \quad \forall x, y \in V, \\ c \odot x &= x^c \quad \forall c \in \mathbb{R}, \text{ and } \forall x \in V. \end{aligned}$$

Prove that V is a vector space over the scalar field $\mathbb{R}$ with the zero vector being 1.

Hint. Although this is a problem in linear algebra, you will need to use something from an earlier topic!

Sketch of solution: Properties 1,3,4,5,6,10 (where the first five concern the operation $\oplus$) are just restatements of the axioms concerning $\mathbb{R}^+$ (since $\mathbb{R}^+ = (0, +\infty)$) and the field axioms associated with multiplication in $\mathbb{R}$, given that we identify

$$V \ni \vec{0} = 1.$$

The other properties are routine consequences of the definition

$$c \odot x \;=\; x^c := e^{c\log(x)}$$

and the properties of exponential function. Let us look at two examples.

- *Property 2* (Closure under scalar multiplication): For any $x \in (0,\infty)$, $c \in \mathbb{R}$, $c \odot x = e^{c\log(x)}$. Since the exponential function is the inverse of log,

$$\mathsf{range}(e^{(\cdot)} = \mathsf{domain}(\log) = (0,\infty).$$

  Thus, $c \odot x \in (0,\infty)$.

- *Property 7* (Associative law for scalar multiplication): For any $x \in (0,\infty)$ and $a, b \in \mathbb{R}$:

$$\begin{aligned} a \odot (b \odot x) = (b \odot x)^a &= (x^b)^a \\ &:= e^{a\log(x^p)} \\ &:= E\big(a\log\big(E(b\log(x))\big)\big) \\ &= E\big(ab\log(x)\big) \\ &=: x^{ab} \\ &= (ab) \odot x, \end{aligned}$$

  where we use E to denote the exponential function.

Since there is **only one tutorial ahead in the semester** it would be good to have some problems from linear algebra for discussion in that tutorial. Thus, following problems will go a little beyond what has been taught until now, and anticipate parts of the lectures of **February 5** and **February 8**.

8. Solve Problems 14, 16, 18, and 20 from Section 15.9 of Apostol, omitting **for the moment** the computation of dimensions.

Sketch of solution: The solutions to Problems 14,16,18, and 20 rely on the theorem that says S is linear subspace if and only if S satisfies the two closure conditions. Based on this, the answers are:

Problem 14: Yes	Problem 16: Yes
Problem 18: Yes	Problem 20: No, unless $k = 0$.

To understand the solution to Problem 20, fix first $k : 0 < k < n$ and consider some $p \in S$ having the following form:

$$p(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0, \quad a_0 \neq 0,$$

where, since $\deg(p) = k$, $a_k \neq 0$. By definition

$$q(x) := -a_k x^k + \sum_{j=0}^{n-1} a_j x^j \in S$$

and $p + q \neq 0$ (this is where the conditions $k > 0$ and $a_0 \neq 0$ are used). However, $\deg(p+q) < k$, so $p + q \notin S$. Since S does not satisfy the closure condition for addition. Thus, S is **not** a linear supspace. Now, this argument does not hold for $k = 0$. In this case, the closure conditions follows from closure axioms of $\mathbb{R}$.

9. By following the reference to the proof of Theorem 12.8 in the discussion of the following result in Apostol, give a proof of the following:

Theorem (THEOREM 15.5 of Apostol)**.** *Let V be a vector space over the field $\mathbb{F}$ and let $\varnothing \neq S \subset V$. Let S have n elements, $n \in \mathbb{N} - \{0\}$. Then any finite subset of $L(S)$ with more than n elements is linearly dependent.*

Sketch of solution: The proof of the stated theorem is exactly the proof of Theorem 12.8 with the following substitutions:

- V_n must be substituted by V.
- The parameter k must be substituted by n.

If we pick a finite set $A \subset L(S)$ with more than n elements, then it would contain a set with $(n+1)$ elements. After the above substitutions, the argument underlying the proof of Theorem 12.8 will show the latter set is linearly dependent. Thus, A is linearly dependent.