UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 11

Instructor: GAUTAM BHARALI **Assigned: FEBRUARY 11, 2021**

---

1. Freely using — without proof — what you know about 3-D coordinate geometry from high school, prove that any plane in $\mathbb{R}^3$ containing the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

2. Let A be some non-empty set and let $V_{\text{fn},\mathbb{R}}(A)$ denote the set of of **all** $\mathbb{R}$-valued functions on A. For any $f, g \in V_{\text{fn},\mathbb{R}}(A)$ and any $c \in \mathbb{R}$, define

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x) \;\; \forall x \in A,\\ (cf)(x) &:= cf(x) \;\; \forall x \in A.\end{aligned}$$

Show that $V_{\text{fn},\mathbb{R}}(A)$ is a vector space over $\mathbb{R}$.

3. Consider the set $S = \{e^{ax}, xe^{ax}\}$, where $a \in \mathbb{R} - \{0\}$, viewed as a subset of $V_{\text{fn},\mathbb{R}}(\mathbb{R})$. Prove that S is a basis of $L(S)$.

4. Let $V_{\text{fn},\mathbb{R}}(\mathbb{R})$ be as in Problem 3. Find the dimension of $L(S)$, $S \subset V_{\text{fn},\mathbb{R}}(\mathbb{R})$, where

a) $S = \{e^x \cos x, e^x \sin x\}$,

b) $S = \{1, \cos 2x, \cos^2 x, \sin^2 x\}$.

5. Let V and W be vector spaces over the field $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Show that T is one-one if and only if $N(T) = \{\vec{0}\}$.

6. Let $\mathcal{P}_n$ denote the vector space of polynomials with real coefficients of degree $\leq n$. Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation given by $T(p) = p''$. Consider the ordered basis $\mathcal{B} = (1, x, \dots, x^n)$. Denote as M the $(n+1) \times (n+1)$-matrix:

$$M = [T]_{\mathcal{B},\mathcal{B}}.$$

Find all the entries M_{ij} of M.

7. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ (although the following makes sense for any field) and consider the linear transformations $T_A : \mathbb{F}^{n_1} \to \mathbb{F}^{n_2}$ and $T_B : \mathbb{F}^{n_2} \to \mathbb{F}^{n_3}$ induced by the matrices A and B, respectively. This means that:

$$\begin{aligned}&A \text{ is an } n_2 \times n_1 \text{ matrix with entries in } \mathbb{F},\\ &B \text{ is an } n_3 \times n_2 \text{ matrix with entries in } \mathbb{F}.\end{aligned}$$

Show that $T_B T_A$ is induced by a matrix C (so, $T_B T_A = T_C$) where $C = BA$ (here, BA is the matrix product that you have learnt in high school).