UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 11 PROBLEMS

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PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. Freely using — without proof — what you know about 3-D coordinate geometry from high school, prove that any plane in $\mathbb{R}^3$ containing the origin $(0,0,0)$ is a subspace of $\mathbb{R}^3$.

Solution: We have learnt that any plane P in $\mathbb{R}^3$ containing the origin $(0,0,0)$ is described in Cartesian coordinates by

$$P = \{(x,y,z) \in \mathbb{R}^3 : Ax + By + Cz = 0\},$$

where $(A,B,C) \neq (0,0,0)$. We now appeal to the theorem that says that P is linear subspace if and only if P satisfies the two closure conditions. To this end, consider $(x,y,z), (x',y',z') \in P$ and $c \in \mathbb{R}$. Observe that

$$A(x+x') + B(y+y') + C(z+z') = (Ax+By+Cz) + (Ax'+By'+Cz') = 0,$$

which implies that $(x,y,z)+(x',y',z') \in P$. Since these two vectors are arbitrary vectors in P, the latter implies that P satisfies the closure condition for addition. Next,

$$A(cx) + B(cy) + C(cz) = c(Ax+By+Cz) = 0,$$

which implies that $c(x,y,z) \in P$. Since (x,y,z) is an arbitrary vector in P and c an arbitrary scalar, the latter implies that P satisfies the closure condition for scalar multiplication. Thus, P is subspace of $\mathbb{R}^3$.

2. Let A be some non-empty set and let $V_{\text{fn},\mathbb{R}}(A)$ denote the set of of **all** $\mathbb{R}$-valued functions on A. For any $f, g \in V_{\text{fn},\mathbb{R}}(A)$ and any $c \in \mathbb{R}$, define

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x) \;\; \forall x \in A,\\ (cf)(x) &:= cf(x) \;\; \forall x \in A.\end{aligned}$$

Show that $V_{\text{fn},\mathbb{R}}(A)$ is a vector space over $\mathbb{R}$.

The above problem involves a **completely routine** check of the ten properties for $V_{\text{fn},\mathbb{R}}(A)$ to be a vector space. Hence, Problem 2 will not be discussed.

3. Consider the set $S = \{e^{ax}, xe^{ax}\}$, where $a \in \mathbb{R} - \{0\}$, viewed as a subset of $V_{\text{fn},\mathbb{R}}(\mathbb{R})$. Prove that S is a basis of $L(S)$.

Solution: Since, by definition, S spans $L(S)$, we must just show that S is linearly independent. We must therefore show that if

$$c_1 e^{ax} + c_2 x e^{ax} = \vec{0},$$

where $\vec{0}$ denotes the zero vector in $V_{\text{fn},\mathbb{R}}(\mathbb{R})$, then $c_1 = c_2 = 0$. The equation above has the meaning

$$c_1 e^{ax} + c_2 x e^{ax} = e^{ax}(c_1 + c_2 x) = 0 \ \ \forall x \in \mathbb{R}. \tag{1}$$

Since we know that $0 \notin \mathsf{range}(e^{a(\cdot)})$, we can cancel e^{ax} on both sides of (1). Hence, we are left **to prove the following:** given

$$c_1 + c_2 x = 0 \ \ \forall x \in \mathbb{R}, \tag{2}$$

we have $c_1 = c_2 = 0$. Now, in particular, the equality in (2) must be satisfied for $x = 0$ and $x = 1$. Substituting $x = 0$ in (2) implies that $c_1 = 0$. Next, substituting $x = 1$ in (2) gives us $c_2 = 0$. Thus, S is linearly independent, and hence a basis.

4. Let $V_{\text{fn},\mathbb{R}}(\mathbb{R})$ be as in Problem 3. Find the dimension of $L(S)$, $S \subset V_{\text{fn},\mathbb{R}}(\mathbb{R})$, where

a) $S = \{e^x \cos x, e^x \sin x\}$,

b) $S = \{1, \cos 2x, \cos^2 x, \sin^2 x\}$.

Sketch of solution: The solution to both Parts (a) and (b) involve producing a finite basis for $L(S)$. By definition, the number of elements in this finte basis will be the dimension of $L(S)$.

a) Show, using an argument similar to that in the solution to Problem 3 that S is a basis of $L(S)$. Then, by definition, the dimension of $L(S)$ is 2.

b) Define $\widetilde{S} := \{\cos^2 x, \sin^2 x\}$. We shall show that $\widetilde{S}$ is a basis of $L(S)$. To this end, consider an arbitrary element $f \in L(S)$. By definition

$$f(x) = a_1 + a_2 \cos 2x + a_3 \cos^2 x + a_4 \sin^2 x = 0 \ \ \forall x \in \mathbb{R}$$

for some $a_1, \dots, a_4 \in \mathbb{R}$. Then, by basic trigonometric identities,

$$f(x) = (a_1 + a_2 + a_3)\cos^2 x + (a_1 - a_2 + a_4)\sin^2 x \ \ \forall x \in \mathbb{R},$$

whence $f \in L(\widetilde{S})$. Since f was an arbitrary element of $L(S)$, we have established that $L(S) \subseteq L(\widetilde{S})$. As $\widetilde{S} \subset S$, by definition $L(\widetilde{S}) \subseteq L(S)$. Hence $L(S) = L(\widetilde{S})$: in other words, $\widetilde{S}$ spans $L(S)$. Now show, using an argument similar to that in the solution to Problem 3, that $\widetilde{S}$ is a basis of $L(S)$. Then, by definition, the dimension of $L(S)$ is 2.

5. Let V and W be vector spaces over the field $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Show that T is one-one if and only if $N(T) = \{\vec{0}\}$.

Solution: First assume that T is one-one. Then, by definition $T^{-1}\{\vec{0}\}$ has at most one element. As T is linear, we have $T(\vec{0}) = \vec{0}$. Thus,

$$N(T) = T^{-1}\{\vec{0}\} = \{\vec{0}\}.$$

Next, assume that $N(T) = \{\vec{0}\}$. Now, consider $x_1, x_2 \in V$ such that $T(x_1) = T(x_2)$. Then

$$\begin{aligned} T(x_1) = T(x_2) &\Rightarrow T(x_1) - T(x_2) = \vec{0} \\ &\Rightarrow T(x_1 - x_2) = \vec{0} && [\text{by linearity}] \\ &\Rightarrow x_1 - x_2 = \vec{0} && [\text{by the definition of } N(T)] \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

Thus, T is one-one.

6. Let $\mathcal{P}_n$ denote the vector space of polynomials with real coefficients of degree $\le n$. Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be the linear transformation given by $T(p) = p''$. Consider the ordered basis $\mathcal{B} = (1, x, \dots, x^n)$. Denote as M the $(n+1) \times (n+1)$-matrix:

$$M \;=\; [T]_{\mathcal{B},\mathcal{B}}.$$

Find all the entries M_{ij} of M.

Sketch of solution: By definition,

$$\text{the } j\text{-th column of } M \;=\; \big[T(j\text{-th term of } \mathcal{B})\big]_{\mathcal{B}}, \quad j = 1, \dots, (n+1). \tag{3}$$

Next, we have

$$T(j\text{-th term of } \mathcal{B}) = T(x^{j-1}) = \begin{cases} 0, & \text{if } j = 1, 2, \\ (j-1)(j-2)x^{j-3}, & \text{if } j \neq 1, 2. \end{cases}$$

Observe that if $n \ge 2$, then x^{j-3} is the element numbered $(j-2)$ in $\mathcal{B}$, $j \neq 1, 2$. Thus, by the above equation and the equation (3)

$$M_{ij} = 0 \;\; \forall (i,j) : \; i = 1, \dots, (n+1), \; j = 1, \dots, (n+1),$$

in case $n = 0, 1$, and

$$\begin{aligned} M_{ij} \;&=\; 0 \quad \text{for each } i, \text{ if } j = 1, 2, \text{ and} \\ M_{ij} \;&=\; \begin{cases} 0, & \text{for } i \neq (j-2), \text{ if } j = 3, \dots, (n+1), \\ (j-1)(j-2), & \text{for } i = (j-2), \text{ if } j = 3, \dots, (n+1), \end{cases} \end{aligned}$$

in case $n \ge 2$.

7. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ (although the following makes sense for any field) and consider the linear transformations $T_A : \mathbb{F}^{n_1} \to \mathbb{F}^{n_2}$ and $T_B : \mathbb{F}^{n_2} \to \mathbb{F}^{n_3}$ induced by the matrices A and B, respectively. This means that:

$$\begin{aligned} &A \text{ is an } n_2 \times n_1 \text{ matrix with entries in } \mathbb{F}, \\ &B \text{ is an } n_3 \times n_2 \text{ matrix with entries in } \mathbb{F}. \end{aligned}$$

Show that $T_B T_A$ is induced by a matrix C (so, $T_B T_A = T_C$) where $C = BA$ (here, BA is the matrix product that you have learnt in high school).

Solution: Fix $(x_1, \dots, x_{n_1}) \in \mathbb{F}^{n_1}$ and write $A = [a_{ij}]$, $B = [b_{ij}]$,

$$\begin{aligned} (y_1, \dots, y_{n_2}) \;&:=\; T_A(x_1, \dots, x_{n_1}), \\ (z_1, \dots, z_{n_3}) \;&:=\; T_B\big(T_A(x_1, \dots, x_{n_1})\big). \end{aligned}$$

By definition

$$z_i = \sum_{k=1}^{n_2} b_{ik}\, y_k = \sum_{j=1}^{n_2} b_{ik} \big(T_A(x_1, \dots, x_{n_1})\big)_k, \quad i = 1, \dots, n_3.$$

We now substitute the definition of T_A into the above equation, to get

$$\begin{aligned} z_i &= \sum_{k=1}^{n_2} b_{ik} \sum_{j=1}^{n_1} a_{kj}\, x_j \\ &= \sum_{j=1}^{n_1} \bigg(\sum_{k=1}^{n_2} a_{ik}\, b_{kj} \bigg) x_j, \quad i = 1, \ldots, n_3. \end{aligned} \tag{4}$$

But since, by definition

$$BA = \bigg[\sum_{k=1}^{n_2} a_{ik}\, b_{kj} \bigg],$$

the equation (4) gives us the i-th Cartesian factor of the vector $T_{BA}(x_1, \ldots, x_{n_1})$. Hence $T_B T_A = T_{BA}$.