UM 101 : ANALYSIS & LINEAR ALGEBRA – I

"AUTUMN" 2020

HOMEWORK 1

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 19, 2020**

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1. In class we encountered one of the axioms of set-theory, stated as "The rule defining when two sets are equal," and for which you were referred to **Section I-2.2** of Apostol's book. Using this rule, justify the following equalities of sets:

(*a*) $\{a, a\} = \{a\}$

(*b*) $\{a, b\} = \{b, a\}$

(*c*) $\{a\} = \{b, c\}$ if and only if $a = b = c$.

2. (Prob. 20(b) from Apostol, Section I-2.5) Show that one of the two expressions below is always right and that the other is sometimes wrong:

$$i)\ A - (B - C) = (A - B) \cup C,$$
$$ii)\ A - (B \cup C) = (A - B) - C.$$

(**Note.** What this means is that you must provide a proof of the expression that you think is always true, and you must provide one counterexample showing that the other is false.)

3. In class, we mentioned that if A and B are two sets, then we take as an axiom — The Axiom of Unions — that $A \cup B$ is a set. In contrast, show that we do not need any axiom beyond those that were mentioned in class to assert that $A \cap B$ is a set. **Specifically** show that the fact that $A \cap B$ is a set is given by the set-builder axiom.

4. Prove that $\varnothing \subseteq A$ for **any** set A.

5. Prove the De Morgan law whose proof was **not** given in class. Namely, if B is a set and $\mathscr{F}$ is a non-empty family of sets, then show that

$$B - \left(\bigcap_{A \in \mathscr{F}} A \right) = \bigcup_{A \in \mathscr{F}} (B - A).$$

The following problem will go a little beyond what has been taught until now. You will need the material of the **lecture of November 20** to solve it.

6. Refer to Peano's Axioms. For a natural number n, $S(n)$ will denote the successor of n. Let "+" denote the Peano addition between two natural numbers (which formalises the addition you learnt as children). Define:

$$1 := S(0),$$
$$2 := S(1) = S(S(0)),$$
$$3 := S(2) = S(S(1)) = S\big(S(S(0))\big).$$

Using the rules of Peano addition, justify that (*a*) $1 + 1 = 2$.

(*b*) $1 + 2 = 3$.

Note. You may freely use the fact $n+m = m+n$ for all $m, n \in \mathbb{N}$ **without proof.** Using this will provide a *somewhat* shorter proof of (*b*) than the one resulting from following the rules of Peano addition slavishly.