UM 101 : ANALYSIS & LINEAR ALGEBRA – I
"AUTUMN" 2020

HINTS/SKETCH OF SOLUTIONS TO HOMEWORK 1 PROBLEMS

Instructor: GAUTAM BHARALI **Assigned: NOVEMBER 19, 2020**

PLEASE NOTE: Only in **rare circumstances** will complete solutions be provided! What follows are **hints** for solving a problem or **sketches** of the solutions meant to help you through the difficult parts (or, sometimes, to introduce a nice trick). You are encouraged to use these to obtain complete solutions.

1. In class we encountered one of the axioms of set-theory, stated as "The rule defining when two sets are equal," and for which you were referred to **Section I-2.2** of Apostol's book. Using this rule, justify the following equalities of sets:

(a) $\{a, a\} = \{a\}$

(b) $\{a, b\} = \{b, a\}$

(c) $\{a\} = \{b, c\}$ if and only if $a = b = c$.

Sketch of solution: We will not discuss all the parts.

(a) $\{a, a\}$ contains the element a and no element other than a. $\{a\}$ contains exactly the element a. Thus, by the definition referenced, $\{a, a\} = \{a\}$.

(b) Use similar reasoning as in part (a).

(c) This problem involves two steps. First assume $\{a\} = \{a, b, c\}$. Then by definition, $b = a$ and $c = a$. Hence $a = b = c$. Argue the implication "$a = b = c \implies \{a\} = \{a, b, c\}$" as done in part (a).

2. (Prob. 20(b) from Apostol, Section I-2.5) Show that one of the two expressions below is always right and that the other is sometimes wrong:

$$i)\ A - (B - C) = (A - B) \cup C,$$
$$ii)\ A - (B \cup C) = (A - B) - C.$$

(**Note.** What this means is that you must provide a proof of the expression that you think is always true, and you must provide one counterexample showing that the other is false.)

Sketch of solution: (i) is false sometimes. Please find specific examples of A, B and C to illustrate this.

(ii) is always true. We first show that $A - (B \cup C) \subseteq (A - B) - C$. Since $\varnothing \subseteq (A - B) - C$, it suffices to consider $A - (B \cup C) \neq \varnothing$. So, pick an arbitrary $x \in A - (B \cup C)$. Then:

$$x \in A \quad \text{and} \quad x \notin (B \cup C). \tag{1}$$

Negating the definition (see Apostol, Section I-2.4) for when an element belongs to $(B \cup C)$, we get

(∗) $x \notin (B \cup C)$ means that x is neither in B nor in C.

From (∗) and (1), we have $x \in A$ and $x \notin B$. So, $x \in (A - B)$. But since $x \notin C$, $x \in (A - B) - C$. As x was arbitrarily chosen, we have $A - (B \cup C) \subseteq (A - B) - C$.

One can prove $(A - B) - C \subseteq A - (B \cup C)$ using (∗). Do this yourself.

3. In class, we mentioned that if A and B are two sets, then we take as an axiom — The Axiom of Unions — that $A \cup B$ is a set. In contrast, show that we do not need any axiom beyond those that were mentioned in class to assert that $A \cap B$ is a set. **Specifically** show that the fact that $A \cap B$ is a set is given by the set-builder axiom.

Sketch of solution: By the set-builder axiom, the collection

$$I = \{x \in A : x \in B\}$$

is a set. Now prove $I = A \cap B$.

4. Prove that $\varnothing \subseteq A$ for **any** set A.

Solution: It is easier to prove this statement by contradiction. So, assume the negation of "$\varnothing \subseteq A$ for any A". Thus, assume there is a set A_0 such that $\varnothing \nsubseteq A_0$. This means that there is an element in $\varnothing$ that is not in A_0. This is a contradiction as $\varnothing$ has no elements.

5. Prove the De Morgan law whose proof was **not** given in class. Namely, if B is a set and $\mathscr{F}$ is a non-empty family of sets, then show that

$$B - \left(\bigcap_{A \in \mathscr{F}} A \right) = \bigcup_{A \in \mathscr{F}} (B - A).$$

Sketch of solution: We will work out one of the inclusions. Let us consider

$$B - \left(\bigcap_{A \in \mathscr{F}} A \right) \subseteq \bigcap_{A \in \mathscr{F}} (B - A).$$

Call the left-hand set C and the right-hand set D. Since $\varnothing \subseteq D$, we may assume $C \neq \varnothing$. Pick an arbitrary element $x \in C$. By definition

$$x \in B \quad \text{and} \quad x \notin \bigcap_{A \in \mathscr{F}} A.$$

By definition of intersection, there is some A in $\mathscr{F}$ such that $x \notin A$. As $x \in B$, $x \in (B - A)$ for some $A \in \mathscr{F}$. By definition of union, therefore $x \in D$. As x was arbitrarily chosen, $C \subseteq D$. The opposite inclusion is left for you to try; it is easy.

The following problem will go a little beyond what has been taught until now. You will need the material of the **lecture of November 20** to solve it.

6. Refer to Peano's Axioms. For a natural number n, $S(n)$ will denote the successor of n. Let "+" denote the Peano addition between two natural numbers (which formalises the addition you learnt as children). Define:

$$\begin{aligned} 1 &:= S(0), \\ 2 &:= S(1) = S(S(0)), \\ 3 &:= S(2) = S(S(1)) = S\big(S(S(0))\big). \end{aligned}$$

Using the rules of Peano addition, justify that

(a) $1 + 1 = 2$.

(b) $1 + 2 = 3$.

Note. You may freely use the fact $n+m = m+n$ for all $m, n \in \mathbb{N}$ **without proof.** Using this will provide a *somewhat* shorter proof of (b) than the one resulting from following the rules of Peano addition slavishly.

Since this involves just applying the rules of Peano addition, the above is left for you to work out.